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CORRIGENDUM

VOLUME 3

G. H. Hardy and **J. E. Littlewood.** *Some integral inequalities connected with the calculus of variations:*

Page 249: lines twelve and thirteen. Instead of

'*with equality only when*

$$y = \eta = Ce^{-x\cos\gamma}\sin(x\sin\gamma - \gamma) \qquad (\gamma = \tfrac{1}{3}\pi)',$$

read

'*with equality only when* $\eta = A\eta(Bx)$, *where*

$$\eta(x) = e^{-x\cos\gamma}\sin(x\sin\gamma - \gamma) \qquad (\gamma = \tfrac{1}{3}\pi)'.$$

The conditions for equality are stated correctly in Theorem 7, but an additional parameter is introduced in the passage from Theorem 7 to Theorem 6 (p. 251, top).

The error was pointed out to us by Professor G. Pólya.

MAXIMA AND MINIMA: THE SEMI-DEFINITE CASE

By T. W. CHAUNDY (*Oxford*)

[Received 17 July 1932]

1. THE theory of the maxima and minima of an analytic* function of a single (real) variable is elementary and complete: we evaluate the sequence of expressions $D^n f(x)$ at the stationary point under review; if the earliest term of the sequence which does not vanish there is positive and of even order, the stationary point is a minimum. This condition is both necessary and sufficient.

When, however, we turn to functions of two or more independent variables, this satisfactory state of affairs ceases. The elementary theory is complete for the case in which the Hessian does not vanish at the stationary point: for the case of the vanishing Hessian it does little more than indicate a procedure.

For instance, with $f(x, y)$, we have, in the neighbourhood of a stationary point,

$$\delta f = f_{xx}(\delta x)^2 + 2f_{xy}\,\delta x \delta y + f_{yy}(\delta y)^2$$

to a first approximation. The elementary theory covers the case in which this approximation is sufficient to determine the sign of δf. If, however, the Hessian

$$H(x, y) \equiv f_{xx} f_{yy} - f_{xy}^2$$

vanishes, the quadratic in δx, δy is semi-definite, i.e. never actually *changes* sign but is zero for a certain value of the ratio $\delta x/\delta y$, and we must then proceed to a closer approximation to decide the sign of δf.

2. If we interpret $f(x, y) =$ constant as a plane curve, the stationary point corresponds to a double point; a maximum or minimum appears as an isolated (conjugate) point, whereas a node with real branches represents a point neither maximum nor minimum. The 'semi-definite' case corresponds to a cusp, and further examination is, in general, necessary to show whether this cusp will appear in the figure as an isolated point or as a double point with real branches.

In practice, methods of successive approximation, such as belong to the technique of curve-tracing, will usually be found adequate.

* We need some condition to ensure that a sufficiency of derivatives exist and do not all vanish. The analytic condition, though too stringent, is convenient.

Scheefer* and Dantscher† have systematized this method of attack, showing, in particular, that the process succeeds in a finite number of steps.

3. In the present note I give explicit criteria for a stationary point to be a minimum,‡ in a form analogous to the elementary criteria for functions of a single variable. That is to say, I define a sequence of expressions $H_n(x, y)$ to be evaluated at the stationary point: at a minimum the first of them which does not vanish must be positive and of even order. Unhappily the sequence is not unique,§ though we may select any preferred variety.

I discuss in detail only the function of two variables. The extension to many variables and to implicit functions is not difficult. I consider only functions which are differentiable as often as we need. I restrict myself naturally to the 'semi-definite' case of vanishing Hessian. I go further and consider only functions whose first approximation at the stationary point is actually quadratic, excluding, for example, such a case as

$$f(x, y) \equiv (a_0, a_1, a_2, a_3, a_4, a_5, a_6)(x, y)^6$$

at the origin. For this confronts us with the problem of determining convenient conditions that the sextic have no real roots, and this is a problem which, *pace* Sturm, is still unmanageable in practice. I confine attention, accordingly, to functions whose critical behaviour depends only on quadratic forms.

For a function $f(x, y)$ I establish the following as my fundamental theorem:

THEOREM 1. *A stationary point at which the Hessian vanishes is a minimum of $f(x, y)$, if and only if the first term of the sequence $H_n(x, y)$ which does not vanish at the point is positive and of even suffix, where*

$$H_n(x, y) \equiv \left\{p\left(f_{xy}\frac{\partial}{\partial x} - f_{xx}\frac{\partial}{\partial y}\right) + q\left(f_{yy}\frac{\partial}{\partial x} - f_{xy}\frac{\partial}{\partial y}\right)\right\}^n f(x, y),$$

p, q being arbitrary parameters independent of x, y.

* *Math. Annalen*, 35 (1890), 541.

† *Math. Annalen*, 42 (1893), 89 and 51 (1899), 227. See also Valiron, *Nouv. Ann.* (4) 20 (1920), 41.

‡ Henceforward I shall disregard maxima as being sufficiently included in the theory of minima.

§ This is not surprising, since we are not interested in H_n unless $H_r = 0$ $(r < n)$. These equations give relations between the derivatives of order less than n, which may be used to modify the form of H_n.

This theorem must be implicit in the theory of Scheefer and Dantscher, since I introduce no new analytical principle. The difference is essentially one of algebraic rearrangement, and for that reason I develop my argument independently.

4. I base my discussion on an examination of $f(x, y)$ along 'curves' $x = x(t)$, $y = y(t)$ through the stationary point: this helps, incidentally, to bring out the analogy with the single-variable theory. It is, of course, clear that if $f(x, y)$ is a minimum at the point, then it is a minimum along every curve through the point, since the whole includes the part. I use the converse of this in the following stricter form:

LEMMA 1. *$f(x, y)$ is a minimum at P, if it is a minimum at P along every curve $x = x(t)$, $y = y(t)$ through P, such that at P*

(i) *the derivatives $D^m x$, $D^m y$ exist for every m not exceeding some specified r;*

(ii) *Dx, Dy are not both zero.*

[Here and afterwards D denotes d/dt.]

The proof depends on showing that, given a set of points having P as a limiting point—here a set of points at which f has a value not exceeding its value at P—we can describe a curve of the type specified in the lemma which will pass through a sub-set of the points, this sub-set also having P as a limiting point. I suppress the proof since the result of the lemma in this more general aspect is not unexpected.*

5. I shall write

$$\partial_x, \partial_y \text{ for } \partial/\partial x, \partial/\partial y, \qquad D_m \text{ for the operator } D^m/m!,$$

$$x_m, y_m \text{ for } D_m x, D_m y, \qquad f_{pq} \text{ for } \partial^{p+q} f/\partial x^p \partial y^q.$$

We are to think of $D_r f$ as a polynomial in $x_1, y_1, \ldots, x_r, y_r$ whose coefficients are partial derivatives of f, and so functions of x, y. These coefficients are to be evaluated at the assigned stationary point (x_0, y_0), and x, y accordingly disappear as variables, unless the contrary is stated. I shall then make frequent use of the following lemma:

* In all probability it is somewhere already in print in some equivalent form.

LEMMA 2. *If we regard $x_1, y_1, \ldots, x_r, y_r$ as all independent, then*

$$\frac{\partial}{\partial x_m}(D_{m+n}f) = D_n f_x \qquad (m+n \leqslant r).$$

To prove this observe that, by Taylor's theorem, $D_{m+n}f$ is the coefficient of t^{m+n} in

$$\sum_{s=0}^{\infty} \{(x_1 t + \ldots + x_r t^r)\partial_x + (y_1 t + \ldots + y_r t^r)\partial_y\}^s f/s!.$$

Differentiation in x_m gives

$$t^m \sum_{s=1}^{\infty} \{(x_1 t + \ldots + x_r t^r)\partial_x + (y_1 t + \ldots + y_r t^r)\partial_y\}^{s-1} \partial_x f/(s-1)!,$$

in which the coefficient of t^{m+n} is $D_n f_x$.

6. It is convenient to impose a homographic transformation on the variables x, y, so that we may have at the stationary point

$$f_{20} = 1, \qquad f_{11} = 0, \qquad f_{02} = 0. \tag{1}$$

The *critical direction* in which alone the semi-definite quadratic

$$D^2 f = f_{20}(Dx)^2 + 2f_{11}\,Dx\,Dy + f_{02}(Dy)^2$$

actually vanishes is now $Dx = 0$. We may define it, in view of (1), as $Df_x = 0$.

More generally we shall have to examine the behaviour of f along *critical curves* whose derivatives satisfy the conditions

$$D_s f_x = 0 \qquad (s = 1, \ldots, r). \tag{2}$$

The highest derivatives involved are x_r, y_r. By Lemma 2

$$\frac{\partial}{\partial x_r}(D_r f_x) = f_{20} = 1, \qquad \frac{\partial}{\partial y_r}(D_r f_x) = f_{11} = 0,$$

so that $D_r f_x = 0$ may be written in the form

$$x_r = A_r(x_{r-1}, \ldots, x_1; y_{r-1}, \ldots, y_1). \tag{3}$$

Hence we may use equations (2) to substitute in succession for $x_r, x_{r-1}, \ldots, x_1$, leaving at length only the variables $y_{r-1}, \ldots, y_1$. The result of such substitution in an expression F I denote by $[F]_r$. The substitution itself I call X_r, and a curve whose derivatives satisfy the relations (2) I call a curve C_r.

7. Now Lemma 2 at once gives

$$\frac{\partial}{\partial y_q}(D_p f_x) = \frac{\partial}{\partial y_{q-m}}(D_{p-m} f_x) \qquad (p \geqslant q \geqslant m),$$

and my first step is to show that this relation persists when we have

applied the respective substitutions X_{p-1}, X_{p-m-1} to the two sides, i.e. to show that

$$\frac{\partial}{\partial y_q}[D_p f_x]_{p-1} = \frac{\partial}{\partial y_{q-m}}[D_{p-m} f_x]_{p-m-1} \qquad (p \geqslant q \geqslant m). \tag{4}$$

Now y_q enters $[F]_r$ both directly and also through every substitution

$$x_s = A_s(x_{s-1}, \ldots, x_1; y_s, \ldots, y_1) \qquad (r \geqslant s \geqslant q),$$

or more precisely through

$$x_s = [A_s]_{s-1}.$$

Thus

$$\frac{\partial}{\partial y_q}[D_p f_x]_{p-1} = \left[\frac{\partial}{\partial y_q}(D_p f_x)\right]_{p-1} + \sum_{s \geqslant q}^{s \leqslant p-1} \left[\frac{\partial}{\partial x_s}(D_p f_x)\right]_{p-1} \frac{\partial}{\partial y_q}[A_s]_{s-1}. \tag{5}$$

If $q = p$, we prove our result (4) at once, for there are now no terms in $\sum$ and we have merely

$$\begin{aligned} \frac{\partial}{\partial y_p}[D_p f_x]_{p-1} &= \left[\frac{\partial}{\partial y_p}(D_p f_x)\right]_{p-1} \\ &= \left[\frac{\partial}{\partial y_{p-m}}(D_{p-m} f_x)\right]_{p-1}, \quad \text{by Lemma 2,} \\ &= \left[\frac{\partial}{\partial y_{p-m}}(D_{p-m} f_x)\right]_{p-m-1}, \end{aligned}$$

since now $p-m-1$ is the highest suffix of x. Thus

$$\frac{\partial}{\partial y_p}[D_p f_x]_{p-1} = \left[\frac{\partial}{\partial y_{p-m}}(D_{p-m} f_x)\right]_{p-m-1}. \tag{6}$$

If $q < p$, I use an inductive argument: fixing q, I show that (5) is true for p, if true for $p-1$. Now by the definition (3)

$$A_s \equiv x_s - D_s f_x,$$

and so

$$\begin{aligned} \frac{\partial}{\partial y_q}[A_s]_{s-1} &= -\frac{\partial}{\partial y_q}[D_s f_x]_{s-1} \\ &= -\frac{\partial}{\partial y_{q-m}}[D_{s-m} f_x]_{s-m-1} \end{aligned} \tag{7}$$

by the hypothesis above, since $s \leqslant p-1$.

Again, in analogy with (6),

$$\left[\frac{\partial}{\partial x_s}(D_p f_x)\right]_{p-1} = \left[\frac{\partial}{\partial x_{s-m}}(D_{p-m} f_x)\right]_{p-m-1}. \tag{8}$$

Thus, by (5), (6), (7), (8),

$$\left[\frac{\partial}{\partial y_q}(D_p f_x)\right]_{p-1} = \left[\frac{\partial}{\partial y_{q-m}}(D_{p-m} f_x)\right]_{p-m-1} - \\ - \sum_{s-m\leqslant p-m-1}^{s-m\geqslant q-m} \left[\frac{\partial}{\partial x_{s-m}}(D_{p-m} f_x)\right]_{p-m-1} \frac{\partial}{\partial y_{q-m}}[D_{s-m} f_x]_{s-m-1} \\ = \left[\frac{\partial}{\partial y_{q-m}}(D_{p-m} f_x)\right]_{p-m-1}.$$

Thus the induction is established, and, since (5) is true for $p = q$, it is proved for any p not less than q.

In similar fashion

$$\frac{\partial}{\partial y_q}[D_p f]_p = \left[\frac{\partial}{\partial y_q}(D_p f)\right]_p + \sum_{s\geqslant q}^{s\leqslant p} \left[\frac{\partial}{\partial x_s}(D_p f)\right]_p \frac{\partial}{\partial y_q}[A_s]_{s-1},$$

and it follows immediately that

$$\frac{\partial}{\partial y_q}[D_p f]_p = \frac{\partial}{\partial y_{q-m}}[D_{p-m} f]_{p-m} \qquad (p \geqslant q \geqslant m). \tag{9}$$

8. Now let us examine the hypothesis that simultaneously

$$\frac{\partial}{\partial y_1}[D_s f]_s = 0 \qquad (s = 1,\ldots, p-1). \tag{10}$$

Then, by (9),

$$\frac{\partial}{\partial y_r}[D_p f]_p = \frac{\partial}{\partial y_1}[D_{p-r+1} f]_{p-r+1} \\ = 0, \text{ by the hypothesis, if } p \geqslant r > 1.$$

Thus $[D_p f]_p$ is a function of y_1 alone. Dimensional considerations show that it is a pure power of y_1: more precisely, we write

$$[D_p f]_p = H_p(x_0, y_0) y_1^p, \tag{11}$$

where H_p involves only partial derivatives of f evaluated at the stationary point (x_0, y_0). Hence, if $H_p(x_0, y_0) = 0$, we shall have

$$[D_p f]_p = 0, \qquad \frac{\partial}{\partial y_1}[D_p f]_p = 0,$$

and we can extend the hypothesis (10) up to $s = p$. More generally, if we have the set of conditions

$$H_r(x_0, y_0) = 0 \quad (r = 1,\ldots, 2n-1), \qquad H_{2n}(x_0, y_0) > 0, \tag{12}$$

we can deduce inductively that

$$[D_r f]_r = 0 \quad (r = 1,\ldots, 2n-1), \qquad [D_{2n} f]_{2n} > 0. \tag{13}$$

9. I now show that the conditions (12) are sufficient to establish a minimum of $f(x, y)$ at the stationary point (x_0, y_0).

In the first place, dimensions show that $D_{2r+1}f$, $D_{2r}f$ are both linear in $x_{r+1},\ldots,x_{2r}$. By Lemma 2

$$\frac{\partial}{\partial x_{r+s}}(D_{2r+1}f) = D_{r-s+1}f_x \qquad (r \geqslant s \geqslant 1).$$

Thus the substitution X_r clears $D_{2r+1}f$ of $x_{r+1},\ldots,x_{2r}$ as well as $x_1,\ldots,x_r$: that is to say,

$$[D_{2r+1}f]_{2r+1} = [D_{2r+1}f]_r.$$

Similarly
$$[D_{2r}f]_{2r} = [D_{2r}f]_r.$$

Moreover, $D_{2r}f$ is quadratic in x_r, and by Lemma 2

$$\frac{\partial}{\partial x_r}(D_{2r}f) = D_rf_x = x_r - A_r, \text{ by (3).}$$

Thus
$$D_{2r}f = \tfrac{1}{2}(x_r - A_r)^2 + B_r,$$

where B_r, like A_r, involves $x_1,\ldots,x_{r-1}$ but not $x_r,\ldots,x_{2r}$, and so

$$[D_{2r}f]_{r-1} = \tfrac{1}{2}\{x_r - [A_r]_{r-1}\}^2 + [B_r]_{r-1}.$$

The further substitution $x_r = [A_r]_{r-1}$ gives

$$[D_{2r}f]_r = [B_r]_{r-1},$$

and so
$$[D_{2r}f]_{r-1} = [D_rf]_r + \tfrac{1}{2}\{x_r - [A_r]_{r-1}\}^2. \qquad (14)$$

Now the curves C_r are a sub-class of the curves C_{r-1}, and (14) shows that, along those curves C_{r-1} that do not also belong to the sub-class C_r,

$$[D_{2r}f]_{r-1} > 0, \text{ if } [D_{2r}f]_r \geqslant 0.$$

If $s \leqslant 2r-1$,

$$[D_sf]_{r-1} = [D_sf]_s = 0, \text{ by (13).}$$

Thus, along a curve C_{r-1} $(r \leqslant n)$ that is not also a curve C_r, the earliest derivative of f which does not vanish at the stationary point is $[D_{2r}f]_{r-1}$, which is positive and of even order: along *all* curves C_n, the earliest derivative which does not vanish at the stationary point is $[D_{2n}f]_n$, which again is positive and of even order.

10. We have now accounted for all curves through the stationary point in which $D^{2n}x$, $D^{2n}y$ exist and Dx, Dy are not both zero. Along all these curves we have shown that $f(x, y)$, under the conditions (13), is a minimum. Hence, according to Lemma 1, conditions (13) suffice to prove $f(x, y)$ a minimum at the stationary point.

Conversely, if along some C_n

$$[D^r f]_n = 0 \qquad (r \leqslant 2n-1),$$

then, since f has to be a minimum along this curve, we must have $[D^{2n} f]_n \geqslant 0$. Hence necessarily for a minimum the earliest $H_r(x_0, y_0)$ which does not vanish must be positive and of even suffix.

We have therefore proved the first half of Theorem 1 by establishing the characteristic sequence H_n. We have still to consider the explicit determination of $H_n(x_0, y_0)$.

11. We no longer restrict ourselves to the stationary point (x_0, y_0) but consider the general functions $H_n(x, y)$. The substitution $x_r = A_r$ appropriate to the stationary point must accordingly be replaced by its complete form

$$D_r(f_x) \equiv f_{20} x_r + f_{11} y_r - A_r = 0,$$

or, let us say,

$$D_r(f_x) \equiv f_{20}(x_r - C_r), \tag{15}$$

where

$$C_r = C_r(x_{r-1}, \ldots, x_1; y_r, \ldots, y_1; x, y).$$

The substitution in this complete form is, I shall show, unaffected by differentiation. The differential operator itself,

$$D \equiv x_1 \partial_x + y_1 \partial_y,$$

admits, of course, the substitution X_1. For brevity I shall denote the transformed operator by D_1, i.e.

$$D_1 \equiv C_1 \partial_x + y_1 \partial_y.$$

I then prove that, for any polynomial ϕ,

$$[D\phi(x_r, x_{r-1}, \ldots)]_{r+1} = D_1\{[\phi(x_r, x_{r-1}, \ldots)]_r\}. \tag{16}$$

The left-hand side of (16) is*

$$[(r+1)x_{r+1}\phi_1 + r x_r \phi_2 + \ldots]_{r+1} = (r+1)[C_{r+1}]_r [\phi_1]_r + r[C_r]_{r-1}[\phi_2]_r + \ldots.$$

The right-hand side is

$$D_1 \phi\{[C_r]_{r-1}, [C_{r-1}]_{r-2}, \ldots\} = D_1\{[C_r]_{r-1}\} \cdot [\phi_1]_r + D_1\{[C_{r-1}]_{r-2}\} \cdot [\phi_2]_r + \ldots.$$

I have then to prove that

$$(s+1)[C_{s+1}]_s = D_1\{[C_s]_{s-1}\} \tag{17}$$

for $s = r, r-1, \ldots$. I prove it inductively.

Assuming then that (16) holds up to $r-1$, we have in particular

$$[DC_r(x_{r-1}, x_{r-2}, \ldots)]_r = D_1\{[C_r(x_{r-1}, x_{r-2}, \ldots)]_{r-1}\},$$

i.e. briefly

$$[DC_r]_r = D_1\{[C_r]_{r-1}\}. \tag{18}$$

* I write ϕ_1 for the partial derivative of ϕ in its first argument, and so on.

Now, from (15),

$$(r+1)f_{20}(x_{r+1}-C_{r+1}) = D\{f_{20}(x_r-C_r)\}$$
$$= f_{20}\{(r+1)x_{r+1}-DC_r\}+(x_r-C_r)Df_{20}.$$

The substitution X_r gives

$$(r+1)[C_{r+1}]_r = [DC_r]_r,$$

i.e., with (18),
$$(r+1)[C_{r+1}]_r = D_1\{[C_r]_{r-1}\}.$$

Hence, if (17) hold up to $s = r-1$, it holds for $s = r$ also, and the induction is established. Since (16) obviously holds for $r = 0$, then (17) holds for $s = 0$ and I have my starting-point and (16) is proved universally.

12. Applying this result to $D_n f$, we have that

$$(n+1)H_{n+1}y_1^{n+1} = (n+1)[D_{n+1}f]_{n+1}$$
$$= D_1\{[D_n f]_n\} = D_1(H_n y_1^n).$$

Now $D(f_x) = f_{20}x_1+f_{11}y_1 = f_{20}(x_1-C_1)$, by definition of C_1. Thus we have, for D_1,

$$D_1 = (y_1/f_{20})(f_{20}\partial_y-f_{11}\partial_x),$$

and so
$$(n+1)H_{n+1}y_1^{n+1} = (y_1/f_{20})(f_{20}\partial_y-f_{11}\partial_x)(H_n y_1^n). \qquad (19)$$

We reach H_{n+1} only when H_n vanishes. The effect of the equation $H_n = 0$ and the definition of H_{n+1} by differentiation of H_n are therefore not affected, if we multiply H_n by any factor which has finite first derivatives at the stationary point, and is not zero or infinite there. It is accordingly sufficient to define the sequence H_n by the recurrence-formula

$$H_{n+1} = (f_{20}\partial_y-f_{11}\partial_x)H_n$$

and so to arrive at the definition

$$H_n = (f_{20}\partial_y-f_{11}\partial_x)^n f, \qquad (20)$$

where after differentiation we put $f_{20} = 1$, $f_{11} = 0$.

13. This is the formula of Theorem 1, but restricted to the special case in which

$$f_{20} = 1, \qquad f_{11} = 0, \qquad f_{02} = 0$$

at the stationary point. To include the general case in which at the stationary point

$$H = 0; \qquad f_{20}, f_{02} \neq 0 \quad (> 0 \text{ for a minimum}),$$

we swing the axes by the linear transformation

$$x = pX+p'Y, \qquad y = qX+q'Y,$$

where p, p', q, q' are constants such that at the stationary point

$$f_{XX} \equiv (p\partial_x+q\partial_y)^2 f = 1, \qquad f_{YY} \equiv (p'\partial_x+q'\partial_y)^2 f = 0,$$
$$f_{XY} \equiv (p\partial_x+q\partial_y)(p'\partial_x+q'\partial_y) f = 0.$$

These equations give at the stationary point

$$f_{20}/q'^2 = -f_{11}/p'q' = f_{02}/p'^2 = 1/(pq'-p'q)^2, \tag{21}$$

and at any point

$$f_{XX}\partial_Y - f_{XY}\partial_X = (p'q-pq')\{p(f_{11}\partial_x-f_{20}\partial_y)+q(f_{02}\partial_x-f_{11}\partial_y)\}.$$

Thus sufficiently we may write

$$H_n \equiv \{p(f_{11}\partial_x-f_{20}\partial_y)+q(f_{02}\partial_x-f_{11}\partial_y)\}^n f. \tag{22}$$

Now equations (21) fix the ratio p'/q', but the ratio p/q is left undefined, provided only that $p'q-pq' \neq 0$. In geometrical language one of the new (X, Y)-axes is fixed as the cuspidal tangent, the other is at our disposal (except, of course, that it must be distinct from the prescribed axis). With this trivial exception we may therefore regard p, q in (22) as 'arbitrary constants' and expand in powers $p^s q^{n-s}$. This gives us $n+1$ coefficients, which are all alternative forms for H_n: they could be proved equivalent by use of the equations $H_r = 0$ $(r < n)$. On the other hand, by giving p, q special pairs of values not proportional to p', q' we obtain infinitely many varieties of H_n, but at most $n+1$ of these varieties are linearly distinct. The proof of Theorem 1 is now complete.

From the theoretical point of view we must remark that the satisfaction of the sequence of conditions for a *particular* p, q is *sufficient* for a minimum: the satisfaction of the conditions for *every* p, q $(p'q-pq' \neq 0)$ is *necessary* for a minimum. In particular, then, their satisfaction for *any* p, q secures their satisfaction for *every* p, q. We may also note that the expressions H_{2n}, regarded as polynomials in p, q, must maintain an invariable sign, under the given conditions, for all real p, q: exceptionally they all vanish when $pq'-p'q = 0$.

14. We are left, as we have said, with a wide variety of choice for H_n. For instance, we may write symmetrically

$$H_n = \{\sqrt{(f_{02})}\partial_x - \sqrt{(f_{20})}\partial_y\}^n f,$$

in which every H of odd order is irrational.

The evaluated forms are not simple: I collect the first three* of them below.

* We ignore H_1, H_2. Actually H_1 is zero, H_2 is the Hessian.

At the stationary point write

$$f_{20} = a^2, \qquad f_{11} = ab, \qquad f_{02} = b^2.$$

Then we have

$$H_3 = (b\partial_x - a\partial_y)^3 f;$$

$$H_4 = (b\partial_x - a\partial_y)^4 f + 3\{(b\partial_x - a\partial_y) f_{20} \cdot (b\partial_x - a\partial_y) f_{02} - [(b\partial_x - a\partial_y) f_{11}]^2\};$$

$$H_5 = (b\partial_x - a\partial_y)^5 f + 5\{(b\partial_x - a\partial_y)^2 f_{20} \cdot (b\partial_x - a\partial_y) f_{02} - 2(b\partial_x - a\partial_y)^2 f_{11} \cdot (b\partial_x - a\partial_y) f_{11} + (b\partial_x - a\partial_y)^3 f_{02} \cdot (b\partial_x - a\partial_y) f_{20}\}.$$

15. The foregoing theory can be extended to functions of more than two independent variables. If, for brevity, we write H_n of Theorem 1 in the symbolic form

$$H_n = \begin{vmatrix} \partial_x & \partial_y & 0 \\ f_{xx} & f_{xy} & -q \\ f_{xy} & f_{yy} & p \end{vmatrix}^n f$$

with the convention that the operators ∂_x, ∂_y have no effect on the other elements of the determinant, then the analogous form for three independent variables x, y, z is found to be

$$H_n = \begin{vmatrix} \partial_x & \partial_y & \partial_z & 0 \\ f_{xx} & f_{xy} & f_{xz} & p \\ f_{xy} & f_{yy} & f_{yz} & q \\ f_{xz} & f_{yz} & f_{zz} & r \end{vmatrix}^n f,$$

where p, q, r are arbitrary constants, and so forth.

Here again the theory is effective only when δf is quadratic at the stationary point. It fails, if every first minor of the Hessian vanishes at the stationary point—as is otherwise obvious, for then every H_n vanishes identically.

16. The theory extends also to functions defined implicitly. Thus, for the minima of $f(x, y, z)$ under the condition $\phi(x, y, z) = 0$, we find that, symbolically,

$$H_n = \begin{vmatrix} \partial_x & \partial_y & \partial_z & 0 & 0 \\ f_{xx} - \lambda\phi_{xx} & f_{xy} - \lambda\phi_{xy} & f_{xz} - \lambda\phi_{xz} & \phi_x & p \\ f_{xy} - \lambda\phi_{xy} & f_{yy} - \lambda\phi_{yy} & f_{yz} - \lambda\phi_{yz} & \phi_y & q \\ f_{xz} - \lambda\phi_{xz} & f_{yz} - \lambda\phi_{yz} & f_{zz} - \lambda\phi_{zz} & \phi_z & r \\ \phi_x & \phi_y & \phi_z & 0 & 0 \end{vmatrix}^n f,$$

where λ is any of $f_x/\phi_x, f_y/\phi_y, f_z/\phi_z$, which are all equal at the stationary point.

A CONTRIBUTION TO THE THEORY OF THE STRAIGHT BEAM

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1. THE problem of the straight beam can be stated as follows:

A straight prismatic beam (see Fig. 1) is subjected to the action of

(i) its own weight, the components of which are

$$X = -\gamma \sin \alpha, \qquad Y = -\gamma \cos \alpha \tag{1}$$

per unit volume, γ being the specific weight of its material;

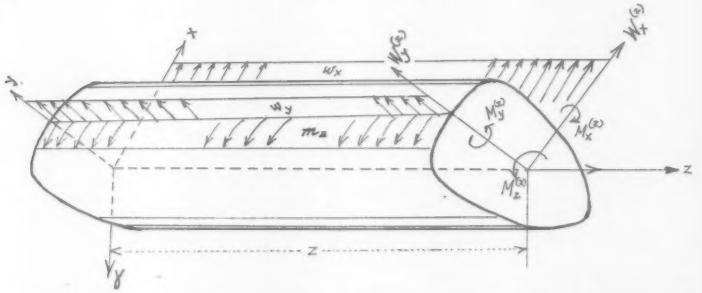

FIG. 1. W_x and W_y are the x and y components of the resultant of all surface forces acting on the prismatic boundary-surface, the distribution of which is not shown in the figure.

(ii) such given surface stresses X_ν, Y_ν, Z_ν as have the following resultant forces and moments:

(a) on the terminal section $z = 0$,
the longitudinal force $N_z^{(0)}$, the transverse forces $W_x^{(0)}$, $W_y^{(0)}$, the bending moments $M_x^{(0)}$, $M_y^{(0)}$, the torsional moment $M_z^{(0)}$;

(b) on the prismatic boundary surface,
the longitudinal force W_z per unit length, transverse forces W_x and W_y per unit length, torsional moment M_z per unit length;

(c) on the terminal section $z = z$,
forces and moments analogous to (a).

The beam is in equilibrium. This means that the forces and moments under (*c*) can be calculated from the rest. The problem is to find the deformation of and the stresses in the beam.

This problem was solved with regard to the action of forces and moments specified under (ii) (*a*) by St. Venant.* We therefore need not consider this part of it here. With regard to the action of forces and moments specified under (i) and (ii) (*b*) the problem was attacked by J. H. Michell† for the case when W_z is equal to zero and W_x, W_y, and M_z are independent of z, i.e. for loads normal to and uniformly distributed along the axis of the beam (but of arbitrary distribution over its boundary). He did not carry it to explicit solution, but only reduced it to a problem of plane strain *with body-forces*. An explicit solution was derived independently, on the semi-inverse method, by the present author‡ in respect of the action of M_z. In the present paper an explicit solution of the 'Michell problem' is given in respect of the action of W_x and W_y by reducing it to a problem of plane strain *without body-forces*.

2. Michell had assumed the stresses to be quadratic functions of the z-coordinates, i.e.

$$\left.\begin{aligned} X_x &= X_x^{(0)}+X_x^{(1)}z+X_x^{(2)}z^2 \\ &\cdots\cdots\cdots \\ Y_z &= Y_x^{(0)}+Y_x^{(1)}z+Y_x^{(2)}z^2 \\ &\cdots\cdots\cdots \end{aligned}\right\}, \qquad (2)\S$$

where $X_x^{(0)}$, $X_x^{(1)}$, $X_x^{(2)}$... are functions of x and y only.

Because of the strain-stress relations

$$\left.\begin{aligned} e_{xx} &= \frac{1}{E}\{X_x-\sigma(Y_y+Z_z)\} \\ &\cdots\cdots\cdots \\ e_{yz} &= \frac{1}{\mu}Y_z \\ &\cdots \end{aligned}\right\} \qquad (3)$$

* *Mém. des savants étrangers*, 14 (1855); *J. de Math.* (Liouville) (2), 8 (1863).

† *Quart. J. of Math.* 32 (1901), 28–42.

‡ *Zeitschr. f. angew. Math. u. Mech.* 5 (1925), 409.

§ The notation follows that used by A. E. H. Love in the *Treatise on the Mathematical Theory of Elasticity*.

we have

$$\left.\begin{aligned} e_{xx} &= e_{xx}^{(0)}+e_{xx}^{(1)}z+e_{xx}^{(2)}z^2 \\ &\cdots\cdots\cdots \\ e_{yz} &= e_{yz}^{(0)}+e_{yz}^{(1)}z+e_{yz}^{(2)}z^2 \\ &\cdots\cdots\cdots \end{aligned}\right\}, \tag{4}$$

i.e. quadratic functions as well, where

$$\left.\begin{aligned} e_{xx}^{(0)} &= \frac{1}{E}\{X_x^{(0)}-\sigma(Y_y^{(0)}+Z_z^{(0)})\} \\ &\cdots\cdots\cdots \\ e_{yz}^{(0)} &= \frac{1}{\mu}Y_z^{(0)} \\ &\cdots\cdots \end{aligned}\right\}. \tag{5}$$

Let us now assume the x and y axes of the coordinate-system to be principal axes of inertia of the cross-section, i.e.

$$\iint x\,dxdy = \iint y\,dxdy = \iint xy\,dxdy = 0. \tag{6}$$

Then Michell derived the following expressions for the deformation-components

$$\left.\begin{aligned} e_{zz}^{(2)} &= -\kappa_2 x-\kappa_2' y \\ e_{xx}^{(2)} &= e_{yy}^{(2)} = -\sigma e_{zz}^{(2)} \\ e_{xy}^{(2)} &= e_{xz}^{(2)} = e_{yz}^{(2)} = 0 \end{aligned}\right\}, \tag{7}$$

$$\left.\begin{aligned} e_{xx}^{(1)} &= e_{yy}^{(1)} = e_{zz}^{(1)} = e_{xy}^{(1)} = 0 \\ e_{zx}^{(1)} &= \tau_1\left(\frac{\partial\phi}{\partial x}-y\right)+2\kappa_2\left\{\frac{\partial\chi}{\partial x}+\tfrac{1}{2}\sigma x^2+(1-\tfrac{1}{2}\sigma)y^2\right\}+ \\ &\qquad +2\kappa_2'\left\{\frac{\partial\chi'}{\partial x}+(2+\sigma)xy\right\} \\ e_{yz}^{(1)} &= \tau_1\left(\frac{\partial\phi}{\partial y}+x\right)+2\kappa_2\left\{\frac{\partial\chi}{\partial y}+(2+\sigma)xy\right\}+ \\ &\qquad +2\kappa_2'\left\{\frac{\partial\chi'}{\partial y}+\tfrac{1}{2}\sigma y^2+(1-\tfrac{1}{2}\sigma)x^2\right\} \end{aligned}\right\}, \tag{8}*$$

$$\left.\begin{aligned} e_{zz}^{(0)} &= \epsilon_0-\kappa_0 x-\kappa_0' y+2\kappa_2(\chi+xy^2)+2\kappa_2'(\chi'+x^2y)+\tau_1\phi \\ e_{zx}^{(0)} &= e_{yx}^{(0)} = 0 \end{aligned}\right\}, \tag{9}*$$

* These expressions have been specialized to exclude the St. Venant case and the m_z case. The exclusion of the St. Venant case means that the forces and moments acting on the terminal section $z = 0$ and as specified under (a) are assumed to be zero.

where κ_2, κ_2', τ_1, ϵ_0, κ_0, κ_0' are constants, ϕ is the torsion-function, and χ and χ' are the two bending-functions of the cross-section, i.e.

$$\left.\begin{aligned} \nabla^2\phi = 0 \qquad \nabla^2\chi = 0 \qquad \nabla^2\chi' = 0 \\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned}\right\}. \tag{10}$$

For the constants he derived the following expressions:

$$\left.\begin{aligned} \kappa_2 = \frac{W_x - \gamma A \sin\alpha}{2EI_y} \qquad \kappa_2' = \frac{W_y - \gamma A \cos\alpha}{2EI_x} \\ \tau_1 = -2\kappa_2 C_2/C_1 - 2\kappa_2' C_2'/C_1 \end{aligned}\right\}, \tag{11}$$

where

$$\left.\begin{aligned} A = \iint dxdy \qquad I_x = \iint y^2\,dxdy \qquad I_y = \iint x^2\,dxdy \\ C_1 = \iint \left[\left(\frac{\partial\phi}{\partial y} + x\right)x - \left(\frac{\partial\phi}{\partial x} - y\right)y\right] dxdy \\ C_2 = \iint \left[\left\{\frac{\partial\chi}{\partial y} + (2+\sigma)xy\right\}x - \left\{\frac{\partial\chi}{\partial x} + \tfrac{1}{2}\sigma x^2 + (1-\tfrac{1}{2}\sigma)y^2\right\}y\right] dxdy \\ C_2' = \iint \left[\left\{\frac{\partial\chi'}{\partial y} + \tfrac{1}{2}\sigma y^2 + (1-\tfrac{1}{2}\sigma)x^2\right\}x - \left\{\frac{\partial\chi'}{\partial x} + (2+\sigma)xy\right\}y\right] dxdy \end{aligned}\right\}. \tag{12}$$

Michell did not derive explicit expressions for $e^{(0)}_{xx}$, $e^{(0)}_{yy}$, $e^{(0)}_{xy}$, or $X^{(0)}_x$, $Y^{(0)}_y$, $X^{(0)}_y$, nor for ϵ_0, κ_0, κ_0'.

3. Such explicit expressions can be derived as is shown in the Appendix. The results are

$$\left.\begin{aligned} X_x = \frac{\partial^2\psi}{\partial y^2} - \gamma x \sin\alpha - \mu\tau_1(\phi - xy) - \\ \quad - 2\mu\kappa_2\{\chi + \tfrac{1}{6}\sigma x^3 + (1-\tfrac{1}{2}\sigma)xy^2\} - 2\mu\kappa_2'\{\chi' + (1+\tfrac{1}{2}\sigma)(x^2y - \tfrac{1}{3}y^3)\} \\ Y_y = \frac{\partial^2\psi}{\partial x^2} - \gamma y \cos\alpha - \mu\tau_1(\phi + xy) - \\ \quad - 2\mu\kappa_2\{\chi + (1+\tfrac{1}{2}\sigma)(xy^2 - \tfrac{1}{3}x^3)\} - 2\mu\kappa_2'\{\chi' + \tfrac{1}{6}\sigma y^3 + (1-\tfrac{1}{2}\sigma)x^2y\} \\ Z_z = E[\epsilon_0 - (\kappa_0 + \kappa_2 z^2)x - (\kappa_0' + \kappa_2' z^2)y] + \\ \quad + 2\mu[\tau_1\phi + 2\kappa_2(\chi + xy^2) + 2\kappa_2'(\chi' + x^2y)] + \\ \quad + \sigma[\nabla^2\psi + \gamma(x\sin\alpha + y\cos\alpha) + \tfrac{2}{3}\mu(\kappa_2 x^3 + \kappa_2' y^3)] \\ Y_z = \mu\left\{\tau_1\left(\frac{\partial\phi}{\partial y} + x\right) + 2\kappa_2\left[\frac{\partial\chi}{\partial y} + (2+\sigma)xy\right] + \right. \\ \quad \left. + 2\kappa_2'\left[\frac{\partial\chi'}{\partial y} + \tfrac{1}{2}\sigma y^2 + (1-\tfrac{1}{2}\sigma)x^2\right]\right\} z \end{aligned}\right\}, \tag{13}$$

$$\left.\begin{aligned} Z_x &= \\ &\mu\left(\tau_1\left(\frac{\partial\phi}{\partial x}-y\right)+2\kappa_2\left[\frac{\partial\chi}{\partial x}+\tfrac{1}{2}\sigma x^2+(1-\tfrac{1}{2}\sigma)y^2\right]+2\kappa_2'\left[\frac{\partial\chi'}{\partial x}+(2+\sigma)xy\right]\right)z \\ X_y &= -\frac{\partial^2\psi}{\partial x\partial y} \end{aligned}\right\}$$

where ψ is the Airy-function which has to satisfy the partial differential equation

$$\nabla^2(\nabla^2\psi) = 0. \tag{14}$$

The constants ϵ_0, κ_0, and κ_0' can be obtained by making $\iint Z_z\,dxdy$, $\iint Z_z x\,dxdy$, $\iint Z_z y\,dxdy$ vanish for $z = 0$. The results are:

$$\left.\begin{aligned} \epsilon_0 &= -\frac{1}{EA}\Big[\sigma\iint\nabla^2\psi\,dxdy+ \\ &\quad +2\mu\Big(\tau_1\iint\phi\,dxdy+2\kappa_2\iint\chi\,dxdy+2\kappa_2'\iint\chi'\,dxdy\Big)\Big] \\ \kappa_0 &= \frac{1}{EI_y}\Big[\sigma\iint\nabla^2\psi x\,dxdy+ \\ &\quad +2\mu\Big(\tau_1\iint\phi x\,dxdy+2\kappa_2\iint\chi x\,dxdy+2\kappa_2'\iint\chi' x\,dxdy\Big)+ \\ &\qquad +\sigma\gamma\sin\alpha\,I_y+4\kappa_2\mu I_{x^2y^2}+\tfrac{2}{3}\sigma\mu\kappa_2 I_{x^4}\Big] \\ \kappa_0' &= \frac{1}{EI_x}\Big[\sigma\iint\nabla^2\psi y\,dxdy+ \\ &\quad +2\mu\Big(\tau_1\iint\phi y\,dxdy+2\kappa_2\iint\chi y\,dxdy+2\kappa_2'\iint\chi' y\,dxdy\Big)+ \\ &\qquad +\sigma\gamma\cos\alpha\,I_x+4\kappa_2'\mu I_{x^2y^2}+\tfrac{2}{3}\sigma\mu\kappa_2' I_{y^4}\Big] \end{aligned}\right\}, \tag{15}$$

where

$$I_{x^2y^2} = \iint x^2y^2\,dxdy, \quad I_{x^4} = \iint x^4\,dxdy, \quad I_{y^4} = \iint y^4\,dxdy. \tag{16}$$

The strain-components can be easily calculated from the foregoing expressions.

The boundary conditions for ϕ, χ, χ' are those of the St. Venant solution. The boundary conditions for ψ can be derived from the expressions

$$\left.\begin{aligned} X_\nu &= X_x\cos(x,\nu)+X_y\sin(x,\nu) \\ Y_\nu &= X_y\cos(x,\nu)+Y_y\sin(x,\nu) \end{aligned}\right\}, \tag{17}$$

where X_ν and Y_ν are the given stresses acting on the prismatic boundary, so that

$$\int X_\nu\,ds = W_x, \qquad \int Y_\nu\,ds = W_y. \tag{18}$$

The Michell problem is thereby reduced to the problem of finding the function ψ from the differential equation (14) and according to the boundary-conditions (17), i.e. to a problem of plane strain without body-forces.

The functions ϕ and χ are known for a number of cross-sections, e.g. the circle, the ellipse, and the rectangle; and by assuming special solutions of equation (14) it should not be difficult to derive expressions for the stresses and deformation of beams of such sections under different loadings. It might be mentioned that for such sections, or generally sections symmetrical about the x and y axes, C_2 and C_2' are equal to zero and therefore also τ_1. Furthermore, for such sections the following expressions are also equal to zero:

$$\iint \phi\, dxdy, \qquad \iint \phi x\, dxdy, \qquad \iint \phi y\, dxdy, \qquad \iint \chi\, dxdy,$$

$$\iint \chi'\, dxdy, \qquad \iint \chi y\, dxdy, \qquad \iint \chi' x\, dxdy.$$

This simplifies the above equations considerably.

Appendix

We start from the following equations:

the equations of equilibrium

$$\left.\begin{aligned} \frac{\partial X_x^{(0)}}{\partial x} + \frac{\partial X_y^{(0)}}{\partial y} + Z_x^{(1)} - \gamma \sin\alpha = 0 \\ \frac{\partial Y_y^{(0)}}{\partial y} + \frac{\partial X_y^{(0)}}{\partial x} + Y_z^{(1)} - \gamma \cos\alpha = 0 \end{aligned}\right\}, \tag{19}$$

and the condition of compatibility

$$\frac{\partial^2 e_{xx}^{(0)}}{\partial y^2} + \frac{\partial^2 e_{yy}^{(0)}}{\partial x^2} = \frac{\partial^2 e_{xy}^{(0)}}{\partial x \partial y}. \tag{20}$$

We introduce into equations (19) the expressions for $Z_x^{(1)}$ and $Y_z^{(1)}$ from

$$Z_x^{(1)} = \mu e_{zx}^{(1)}, \qquad Y_z^{(1)} = \mu e_{yz}^{(1)}, \tag{21}$$

taking $e_{zx}^{(1)}$ and $e_{yz}^{(1)}$ from equations (8).

We get accordingly

$$\left.\begin{aligned} \frac{\partial X_x^{(0)}}{\partial x} + \frac{\partial X_y^{(0)}}{\partial y} - \gamma \cos\alpha + \mu\tau_1 \left(\frac{\partial \phi}{\partial x} - y\right) + \\ + 2\mu\kappa_2 \left\{\frac{\partial \chi}{\partial x} + \tfrac{1}{2}\sigma x^2 + (1 - \tfrac{1}{2}\sigma) y^2\right\} + 2\mu\kappa_2' \left\{\frac{\partial \chi'}{\partial x} + (2+\sigma) xy\right\} = 0 \end{aligned}\right\}. \tag{22}$$

$$\left.\begin{aligned}&\frac{\partial Y_y^{(0)}}{\partial y}+\frac{\partial X_y^{(0)}}{\partial x}-\gamma\sin\alpha+\mu\tau_1\left(\frac{\partial\phi}{\partial y}+x\right)+\\&\quad+2\mu\kappa_2\left(\frac{\partial\chi}{\partial y}+(2+\sigma)xy\right)+2\mu\kappa_2'\left(\frac{\partial\chi'}{\partial y}+\tfrac{1}{2}\sigma y^2+(1-\tfrac{1}{2}\sigma)x^2\right)=0\end{aligned}\right|$$

Now we introduce into equation (20) expressions for the stresses, by considering equations (5), and get

$$\frac{\partial^2 X_x^{(0)}}{\partial y^2}+\frac{\partial^2 Y_y^{(0)}}{\partial x^2}-\sigma\left(\frac{\partial^2 X_x^{(0)}}{\partial x^2}+\frac{\partial^2 Y_y^{(0)}}{\partial y^2}+\nabla^2 Z_z^{(0)}\right)=\frac{E}{\mu}\frac{\partial^2 X_y^{(0)}}{\partial x\partial y}. \tag{23}$$

Furthermore, there follows from the first of equations (5)

$$Z_z^{(0)}=E[\epsilon_0-\kappa_0 x-\kappa_0' y+2\kappa_2(\chi+xy^2)+2\kappa_2'(\chi'+x^2y)+\tau_1\phi]+\\+\sigma(X_x^{(0)}+Y_y^{(0)}). \tag{24}$$

Introducing (24) into (23), and considering equations (10), we get

$$\frac{\partial^2 X_x^{(0)}}{\partial y^2}+\frac{\partial^2 Y_y^{(0)}}{\partial x^2}-\sigma\left(\frac{\partial^2 X_x^{(0)}}{\partial x^2}+\frac{\partial^2 Y_y^{(0)}}{\partial y^2}\right)-4E\sigma(\kappa_2 x+\kappa_2' y)-\\-\sigma^2\nabla^2(X_x^{(0)}+Y_y^{(0)})=\frac{E}{\mu}\frac{\partial^2 X_x^{(0)}}{\partial x\partial y}. \tag{25}$$

Now we differentiate the first of equations (22) in respect to x and the second in respect to y, and get

$$\left.\begin{aligned}&\frac{\partial^2 X_x^{(0)}}{\partial x^2}+\frac{\partial^2 X_y^{(0)}}{\partial x\partial y}+\mu\tau_1\frac{\partial^2\phi}{\partial x^2}+2\mu\kappa_2\left(\frac{\partial^2\chi}{\partial x^2}+\sigma x\right)+\\&\qquad+2\mu\kappa_2'\left(\frac{\partial^2\chi'}{\partial x^2}+(2+\sigma)y\right)=0\\&\frac{\partial^2 Y_y^{(0)}}{\partial y^2}+\frac{\partial^2 X_y^{(0)}}{\partial x\partial y}+\mu\tau_1\frac{\partial^2\phi}{\partial y^2}+2\mu\kappa_2\left(\frac{\partial^2\chi}{\partial y^2}+(2+\sigma)x\right)+\\&\qquad+2\mu\kappa_2'\left(\frac{\partial^2\chi'}{\partial y^2}+\sigma y\right)=0\end{aligned}\right\}, \tag{26}$$

and, if both these equations are added and again (10) kept in mind, this gives

$$\frac{\partial^2 X_x^{(0)}}{\partial x^2}+\frac{\partial^2 Y_y^{(0)}}{\partial y^2}+2\frac{\partial^2 X_y^{(0)}}{\partial x\partial y}+4\mu(1+\sigma)(\kappa_2 x+\kappa_2' y)=0. \tag{27}$$

By introducing the value for $\dfrac{\partial^2 X_x^{(0)}}{\partial x^2}+\dfrac{\partial^2 Y_y^{(0)}}{\partial y^2}$ out of this equation into equation (25) we get

$$\frac{\partial^2 X_x^{(0)}}{\partial y^2}+\frac{\partial^2 Y_y^{(0)}}{\partial x^2}-2\frac{\partial^2 X_y^{(0)}}{\partial x\partial y}-\sigma\nabla^2(X_x^{(0)}+Y_y^{(0)})-4\mu\sigma(1+\sigma)(\kappa_2' x+\kappa_2 y)=0, \tag{28}$$

since
$$\mu = \frac{E}{2(1+\sigma)}. \tag{29}$$

From equations (27) and (28) there follows

$$\nabla^2(X_x^{(0)}+Y_y^{(0)})-\sigma^2\nabla^2(X_x^{(0)}+Y_y^{(0)})-4\mu(1-\sigma^2)(\kappa_2 x+\kappa_2' y)=0 \tag{30}$$

i.e.
$$\nabla^2(X_x^{(0)}+Y_y^{(0)})+4\mu(\kappa_2 x+\kappa_2' y)=0. \tag{31}$$

Finally equations (28) and (31) give

$$\frac{\partial^2 X_x^{(0)}}{\partial y^2}+\frac{\partial^2 Y_y^{(0)}}{\partial x^2}-2\frac{\partial^2 X_y^{(0)}}{\partial x\partial y}-4\mu\sigma(\kappa_2 x+\kappa_2' y)=0. \tag{32}$$

Now let $X_x^{(0)\prime}$, $Y_y^{(0)\prime}$, and $X_y^{(0)\prime}$ be *particular integrals* of differential equations (22) and (32). Then

$$X_x^{(0)\prime\prime}=X_x^{(0)}-X_x^{(0)\prime}, \quad Y_y^{(0)\prime\prime}=Y_y^{(0)}-Y_y^{(0)\prime}, \quad X_y^{(0)\prime\prime}=X_y^{(0)}-X_y^{(0)\prime} \tag{33}$$

are the stress-components of a certain state. To determine this state we have to consider that, because of the definition of $X_x^{(0)\prime}$, $Y_y^{(0)\prime}$, $X_y^{(0)\prime}$, equations (22) and (32) give

$$\left.\begin{aligned}
&\frac{\partial X_x^{(0)\prime}}{\partial x}+\frac{\partial X_y^{(0)\prime}}{\partial y}-\gamma\cos\alpha+\mu\tau_1\left(\frac{\partial\phi}{\partial x}-y\right)+\\
&\quad+2\mu\kappa_2\left(\frac{\partial\chi}{\partial x}+\tfrac{1}{2}\sigma x^2+(1-\tfrac{1}{2}\sigma)y^2\right)+2\mu\kappa_2'\left(\frac{\partial\chi'}{\partial x}+(2+\sigma)xy\right)=0\\
&\frac{\partial Y_y^{(0)\prime}}{\partial y}+\frac{\partial X_y^{(0)\prime}}{\partial x}-\gamma\sin\alpha+\mu\tau_1\left(\frac{\partial\phi}{\partial y}+x\right)+\\
&\quad+2\mu\kappa_2\left(\frac{\partial\chi}{\partial y}+(2+\sigma)xy\right)+2\mu\kappa_2'\left(\frac{\partial\chi'}{\partial y}+\tfrac{1}{2}\sigma y^2+(1-\tfrac{1}{2}\sigma)x^2\right)=0\\
&\frac{\partial^2 X_x^{(0)\prime}}{\partial y^2}+\frac{\partial^2 Y_y^{(0)\prime}}{\partial x^2}-2\frac{\partial^2 X_y^{(0)\prime}}{\partial x\partial y}-4\mu\sigma(\kappa_2 x+\kappa_2' y)=0
\end{aligned}\right\}. \tag{34}$$

If we now deduce equations (34) from equations (22) and (32) we get

$$\left.\begin{aligned}
&\frac{\partial X_x^{(0)\prime\prime}}{\partial x}+\frac{\partial X_y^{(0)\prime\prime}}{\partial y}=0, \quad \frac{\partial Y_y^{(0)\prime\prime}}{\partial y}+\frac{\partial X_y^{(0)\prime\prime}}{\partial x}=0\\
&\frac{\partial^2 X_x^{(0)\prime\prime}}{\partial y^2}+\frac{\partial^2 Y_y^{(0)\prime\prime}}{\partial x^2}-2\frac{\partial^2 X_y^{(0)\prime\prime}}{\partial x\partial y}=0
\end{aligned}\right\}. \tag{35}$$

This, however, is a problem of plane strain with *no mass-forces*, and equations (35) can therefore be satisfied through the Airy function ψ:

$$\left.\begin{aligned}
&X_x^{(0)\prime\prime}=\frac{\partial^2\psi}{\partial y^2}, \quad Y_y^{(0)\prime\prime}=\frac{\partial^2\psi}{\partial x^2}, \quad X_y^{(0)\prime\prime}=-\frac{\partial^2\psi}{\partial x\partial y}\\
&\nabla^2(\nabla^2\psi)=0
\end{aligned}\right\}. \tag{36}$$

The question therefore is only to find particular integrals of equations (22) and (32). This can be done as follows. From equations (22) we have, through integration,

$$\left.\begin{aligned} X_x^{(0)} + \int \frac{\partial X_y^{(0)}}{\partial y}\,dx - \gamma x \sin\alpha + \mu\tau_1(\phi - xy) + 2\mu\kappa_2\{\chi + \tfrac{1}{6}\sigma x^3 + \\ + (1-\tfrac{1}{2}\sigma)xy^2\} + 2\mu\kappa_2'\{\chi' + (1+\tfrac{1}{2}\sigma)x^2y\} + F_1(y) = 0 \\ Y_y^{(0)} + \int \frac{\partial X_y^{(0)}}{\partial x}\,dy - \gamma y \cos\alpha + \mu\tau_1(\phi + xy) + 2\mu\kappa_2\{\chi + (1+\tfrac{1}{2}\sigma)xy^2\} \\ + 2\mu\kappa_2'\{\chi' + \tfrac{1}{6}\sigma y^3 + (1-\tfrac{1}{2}\sigma)x^2y\} + F_2(x) = 0 \end{aligned}\right\}, \quad (37)$$

where F_1 is a function of y only and F_2 a function of x only. Out of these total integrals, particular integrals can be derived by assuming

$$X_y^{(0)\prime} = 0. \quad (38)$$

If now these particular integrals are introduced into equation (32), this gives

$$2\mu(2+\sigma)(\kappa_2 x + \kappa_2' y) + \frac{d^2F_1}{dy^2} + \frac{d^2F_2}{dx^2} = 0. \quad (39)$$

From this it follows that we can assume

$$\frac{d^2F_1}{dy^2} = -2\mu(2+\sigma)\kappa_2' y, \qquad \frac{d^2F_2}{dx^2} = -2\mu(2+\sigma)\kappa_2 x, \quad (40)$$

or $$F_1(y) = -\tfrac{1}{3}\mu(2+\sigma)\kappa_2' y^3, \qquad F_2(x) = -\tfrac{1}{3}\mu(2+\sigma)\kappa_2 x^3, \quad (41)$$

and our particular integrals are

$$\left.\begin{aligned} X_x^{(0)\prime} = -\gamma x \sin\alpha - \mu\tau_1(\phi - xy) - 2\mu\kappa_2\{\chi + \tfrac{1}{6}\sigma x^3 + \\ + (1-\tfrac{1}{2}\sigma)xy^2\} - 2\mu\kappa_2'\{\chi' + (1+\tfrac{1}{2}\sigma)(x^2y - \tfrac{1}{3}y^3)\} \\ Y_y^{(0)\prime} = -\gamma y \cos\alpha - \mu\tau_1(\phi - xy) - 2\mu\kappa_2\{\chi + (1+\tfrac{1}{2}\sigma)(xy^2 + \tfrac{1}{3}x^3)\} - \\ - 2\mu\kappa_2'\{\chi' + \tfrac{1}{6}\sigma y^3 + (1-\tfrac{1}{2}\sigma)x^2y\} \\ X_y^{(0)\prime} = 0 \end{aligned}\right\}. \quad (42)$$

A CLASS OF TRIANGLE-FUNCTIONS (II)

By J. HODGKINSON (*Oxford*)

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In my earlier paper I discussed the general form of a zeta-Fuchsian triangle-function, i.e. a one-valued function w which undergoes linear transformation when its argument u undergoes the linear transformations associated, in the Schwarzian theory of conformal representation, with a curvilinear triangle of angles π/m, π/n, π/p, where m, n, p are positive integers greater than unity.* In this paper I obtain explicit evaluations of w when the group of u is polyhedral or is associated with the representation of a rectilinear triangle.

It must be acknowledged that the general form discussed in (**I**) is itself an explicit evaluation for the special functions considered, for w is expressed in terms of hypergeometric functions of a variable z,† together with an arbitrary one-valued function of z, and here z is known explicitly in terms of u. This evaluation, however, is needlessly complex, and fails to display, for example, the simple algebraic character of zeta-Fuchsian polyhedral functions.

I shall use the notation $u(\alpha, \beta, \gamma)$, $w(\lambda, \mu, \nu)$ to indicate that I am considering the determination of a function w which undergoes the group of transformations associated with a triangle of angles $\lambda\pi$, $\mu\pi$, $\nu\pi$, when the variable u undergoes the group associated with a triangle of angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$.

I find the following is the list of problems to be considered:

(i) $u(\alpha, \beta, \gamma)$, $w(\alpha, \beta, \gamma)$; all polyhedral cases;

(ii) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$;

(iii) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$, $w(\frac{1}{2}, \frac{1}{3}, \frac{2}{5})$;

(iv) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $w(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$; (iv *a*) $u(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $w(\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$;

(v) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $w(\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$;

(vi) $u(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$; (vi *a*) $u(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$;

(vii) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $w(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$; (vii *a*) $u(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $w(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$;

* J. Hodgkinson, *Quart. J. of Math.* (Oxford), **3** (1932), 142–51. This paper will be referred to as (**I**).

† As in (**I**) I suppose always that the u-triangle has been mapped upon the half-plane of z, with the angular points corresponding to the z-points 0, 1, ∞, so that z is the fundamental function automorphic for the group of transformations of u.

(viii) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $w(\frac{1}{2}, \frac{2}{3}, \frac{1}{6})$; (viii a) $u(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $w(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$;

(ix) $u(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$;

(x) $u(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{3}{4}, \frac{1}{4})$.

In this list the problem marked (a) is really included in the other problem with the same numeral. An equilateral triangle may be built up of two Schwarzian repetitions of a triangle of angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{6}\pi$; an isosceles right-angled triangle similarly may be built up of two repetitions of a smaller isosceles right-angled triangle. The group of u in the problem (a) is a sub-group of the group of u in the other problem, and the groups of w have an exactly corresponding relation. The sole difference in the two problems is in the argument of the arbitrary one-valued function of z required in the construction of the general function w; this, of course, must be the fundamental function automorphic for the group of u.

All other zeta-Fuchsian functions associated with polyhedral or rectilinear triangles may be included in one of the types given as the general arguments of (**I**) indicate.

It is implicit in (**I**), though not clearly shown, that two groups may be associated with a rectilinear configuration but only one with the more general curvilinear configuration. The reason for this is obvious when the rectilinear configuration is inverted with respect to an arbitrary point O.

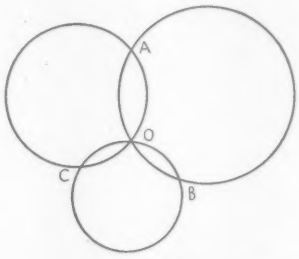


FIG. 1

Whereas in the general case (**I**, p. 147) it is possible to select that one of two triangles of angles A, $\pi - B$, $\pi - C$, whose sides occur in the order BC, CA, AB as the contour is described in the positive direction, no such choice is possible in Fig. 1 above, the plane now being divided into four triangles and three lunes. For such a triangle we use the repetition of a triangle colunar with ABC obtained by

inversion (reflection) in one of its sides. This is shown in the figures of §§ (viii), (x). From our present point of view the two groups are distinct, though consisting of the same substitutions in different orders, and this displays itself in the difference of character of the functions w of (viii), (x) when compared with those of (vii), (ix).

In place of using the general form given in (I), I find it more convenient to revert to the initial stage of that paper, where it was seen that w is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = f(z),$$

where w_1, w_2, w_3 are any three functions admitting the group of transformations required to be associated with w, so that each problem is solved by finding three such functions. I omitted to note in (I) that only two functions are needed when the group of w is associated with a rectilinear triangle. For then every transformation of w is of the form $W = aw+b$, whence it follows that $(w-w_1)/(w_2-w_1) = f(z)$. This applies to §§ (vii)—(x).

(i) The polyhedral cases of $u(\alpha, \beta, \gamma)$, $w(\alpha, \beta, \gamma)$.

I have discussed these elsewhere from a somewhat different aspect.* The simpler method used there, i.e. by differentiation of the covariants of the ground-forms associated with the polyhedral groups, is the more suitable for our present purpose, as it enables us to calculate, not only three, but any number of functions having the required properties.

(ii) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$.

The dihedral group associated with the triangle of angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{2}\pi$ is essentially the anharmonic group, and we accordingly replace w by λ to suggest the latter group. The relation between λ, u occurs frequently in Klein's writings; it is given by him in the form†

$$\lambda = -\frac{(u^2-1)^2}{4u^2}.$$

A proof by the method used in (v) below is very brief.

The unit triangle associated with the anharmonic group has angles of the form π/m, π/n, π/p. Hence functions of λ admitting the same

* J. Hodgkinson, *Proc. London Math. Soc.* (2), 24 (1925), 71–82 (77–8).

† Klein-Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, 1 (1890), 20–1, and 74–6.

group as λ itself can be found by the method of differentiation already referred to. Two ground-forms associated with this group are

$$\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2, \qquad 2\lambda_1^3-3\lambda_1^2\lambda_2-3\lambda_1\lambda_2^2+2\lambda_2^3.$$

From these, we obtain as our second and third functions

$$\frac{\lambda-2}{2\lambda-1}, \qquad \frac{\lambda^2+2\lambda-2}{2\lambda^2-2\lambda-1}.$$

(iii) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}), w(\frac{1}{2}, \frac{1}{3}, \frac{2}{5})$.

The group of w is the icosahedral group but occurs in a different order from that associated with u. We have to determine three functions w_1, w_2, w_3, and we do this most simply from the consideration of a triangle of angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{2}{5}\pi$, and its associated triangles of angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{3}{5}\pi$, and $\frac{1}{2}\pi$, $\frac{2}{3}\pi$, $\frac{2}{5}\pi$. The choice of the first is not unique; it can be made in sixty distinct ways in the same icosahedral configuration; but when this choice has been made the associated triangles must be taken so that they are bounded by the same three circles taken in the same order (**I**, p. 147). Fig. 2 shows the relation of my choice to the conventional icosahedral subdivision of the plane.

I find that

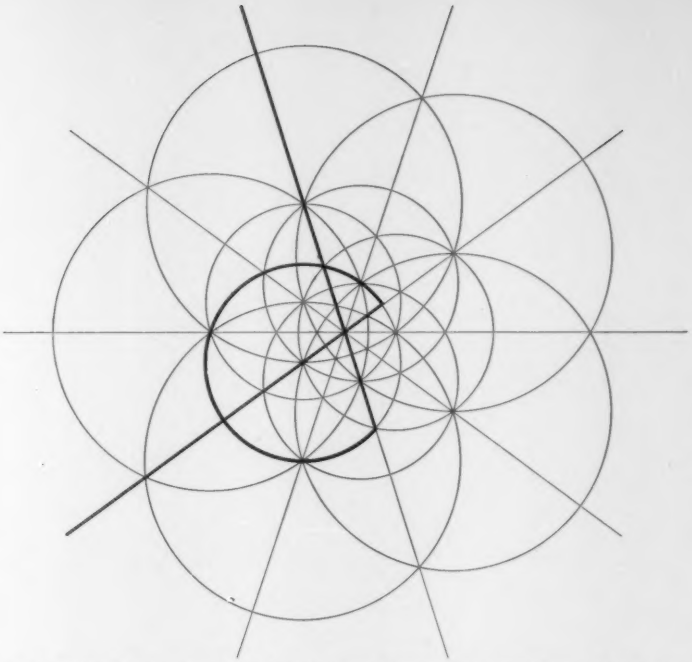
$$w_1 = -\exp(\tfrac{1}{5}i\pi)\frac{u^7-7u^2}{7u^5+1}, \qquad w_2 = \exp(\tfrac{0}{5}i\pi)\frac{26u^{10}-39u^5-1}{u^{13}-39u^8-26u^3},$$

$$w_3 = -\exp(\tfrac{0}{5}i\pi)\frac{u^{17}+119u^{12}+187u^7+17u^2}{17u^{15}-187u^{10}+119u^5-1}.$$

The numerators and denominators of these fractions are to be recognized as forms occurring in the researches of Klein, who points out that they also occur in those of Gordan, on the solution of the equation of the fifth degree. Their connexion with conformal representation was shown by Fischer in a dissertation which I have not been able to consult in its complete form.* I describe here a method which obtains the three functions directly. I consider only the function w_3, omitting the suffix for convenience.

The triangle associated with w contains seventeen repetitions of the unit triangle of the configuration. If u is subjected to the full group of icosahedral transformations, so that the whole plane of u is

* F. Klein, *Math. Annalen*, 12 (1877), 502–60; *Vorlesungen über das Ikosaeder* (1884), 189, 196, Klein-Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, 2 (1892), 385–6; P. Gordan, *Math. Annalen* (1878), 375–404; O. Fischer, *Leipziger Berichte*, 36 (1884), 17–31.



The darkened lines show the boundaries of the triangles; the figures below show which are the three triangles chosen and the positive sense of each contour.

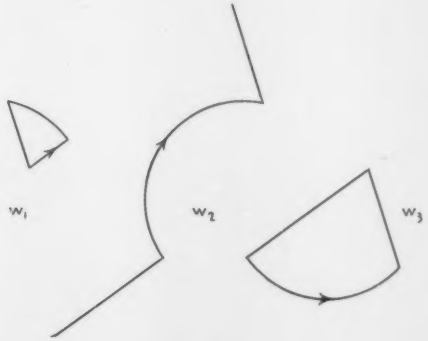


FIG. 2

covered completely by the repetitions of the unit figure, the plane of w is covered seventeen times. Hence the relation between w, u is of the seventeenth degree in u.

An angle $\frac{1}{5}\pi$ at the origin in the u-figure is transformed into an angle $\frac{2}{5}\pi$ at the origin in the w-figure. Hence $u = 0$ is a double zero of w. The transformation $w' = w\exp(\frac{1}{5}i\pi)$ corresponds to the transformation $u' = u\exp(\frac{2}{5}i\pi)$. Hence

$$w = u^2F(u^5).$$

As u moves from the origin along the side of the initial u-triangle which coincides with the real axis, $\arg w = \frac{6}{5}\pi$, whence

$$w = \exp(\tfrac{6}{5}i\pi)\frac{a_0u^{17}+a_1u^{12}+a_2u^7+a_3u^2}{b_0u^{15}+b_1u^{10}+b_2u^5+b_3},$$

and a_3/b_3 is positive.

As u tends to infinity, w tends to infinity (for a_0 cannot vanish if the required degree of the relation is to be preserved). Thus there must be an icosahedral transformation of the form $w' = -\exp(\frac{2}{5}ki\pi)/w$ corresponding to the transformation $u' = -1/u$. If, then, the zeros of w are given by $u^5 = \alpha^5, \beta^5, \gamma^5$, the poles of w are given by $u^5 = -1/\alpha^5$, $-1/\beta^5$, $-1/\gamma^5$. But by considering a suitable succession of repetitions of the initial u-triangle and the corresponding repetitions of the initial w-triangle, it is found that w has three negative zeros (shown in Fig. 3 in the 4th, 5th, 6th repetitions), and that, as u tends to infinity in the negative direction, w tends to infinity in the direction $\arg w = \frac{1}{5}\pi$; hence, if $u' = -1/u$, $w' = -1/w$.

From these considerations we can write

$$w = -\exp(\tfrac{6}{5}i\pi)\frac{u^{17}+\lambda u^{12}+\mu u^7+\nu u^2}{\nu u^{15}-\mu u^{10}+\lambda u^5-1},$$

where λ, μ, ν are positive.

Now the effect of applying to u the transformation which Klein denotes by $T(u)$ is to apply to w a transformation $T'(w)$. $T(u)$ is equivalent to successive inversions in those sides of the initial u-triangle which contain the right angle; similarly for $T'(w)$. $T(u)$ transforms 0 into α, α into 0, ∞ into $-1/\alpha$, $-1/\alpha$ into ∞ (three of these suffice to determine T) where $\alpha = \frac{1}{2}(\sqrt{5}-1)$;* i.e.

$$T(u) \equiv \frac{-u+\alpha}{\alpha u+1}.$$

* The conventional scales are used for the polyhedral configurations throughout the paper.

$T'(w)$ transforms 0 into $\exp(\frac{1}{5}i\pi)/\alpha$, $\exp(\frac{1}{5}i\pi)/\alpha$ into 0, ∞ into $\alpha\exp(\frac{6}{5}i\pi)$, and $\alpha\exp(\frac{6}{5}i\pi)$ into ∞; i.e.

$$T'(w) \equiv \exp(\tfrac{1}{5}i\pi)\frac{-\alpha w+\exp(\frac{1}{5}i\pi)}{w+\alpha\exp(\frac{1}{5}i\pi)}.$$

The figure shows the initial u-triangle and six repetitions; the seven triangles are supposed numbered from the right for comparison with the w-figure below.*

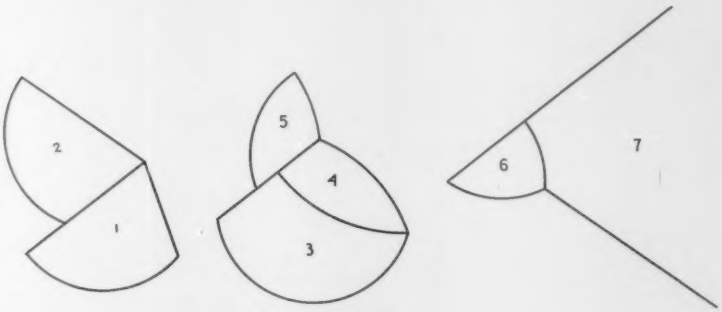


FIG. 3

If now we replace u in the expression for w by $T(u)$, we obtain $T'(w)$, and we find, after reduction, that we have to determine λ, μ, ν so that the equation

$$\frac{(u-\alpha)^{17}-\lambda(u-\alpha)^{12}(\alpha u+1)^5+\mu(u-\alpha)^7(\alpha u+1)^{10}-\nu(u-\alpha)^2(\alpha u+1)^{15}}{\nu(u-\alpha)^{15}(\alpha u+1)^2+\mu(u-\alpha)^{10}(\alpha u+1)^7+\lambda(u-\alpha)^5(\alpha u+1)^{12}+(\alpha u+1)^{17}}$$
$$=\frac{-\alpha u^2(u^{15}+\lambda u^{10}+\mu u^5+\nu)+(\nu u^{15}-\mu u^{10}+\lambda u^5-1)}{u^2(u^{15}+\lambda u^{10}+\mu u^5+\nu)+\alpha(\nu u^{15}-\mu u^{10}+\lambda u^5-1)}$$

is an identity.

* It is impossible to put the seven w-triangles into one figure on account of overlapping. The size and orientation of the triangles enables their position in the icosahedral configuration to be determined.

Clearly $u = \alpha$ must be a double zero of the numerator of the right-hand member of this equation. Hence

$$\lambda(\alpha^5-\alpha^{13})-\mu(\alpha^8+\alpha^{10})-\nu(\alpha^3-\alpha^{15}) = 1+\alpha^{18},$$

and $$\lambda(5\alpha^2-12\alpha^{10})-\mu(7\alpha^5+10\alpha^7)-\nu(2-15\alpha^{12}) = 17\alpha^{15}.$$

After substitution of the value of α, we find that

$$3\lambda-\mu-8\nu = 34$$

and

$$\lambda(655\sqrt{5}-1461)-\mu(165\sqrt{5}-367)-\nu(2160\sqrt{5}-4826) = 34(305\sqrt{5}-682).$$

Further, the coefficient of u^{14} in the numerator of the left-hand member must vanish; i.e.

$$2\lambda(1-12\alpha^2+33\alpha^4-22\alpha^6)-\mu(22\alpha^5-63\alpha^7+42\alpha^9-7\alpha^{11})+ +\nu(91\alpha^{10}-42\alpha^{12}+3\alpha^{14}) = -136\alpha,$$

or $$(89\sqrt{5}-119)\lambda-(53\sqrt{5}-118)\mu-(44\sqrt{5}-99)\nu = -68(\sqrt{5}-1).$$

Hence we find that $\lambda = 119$, $\mu = 187$, $\nu = 17$, so that

$$w = -\exp(\tfrac{6}{5}i\pi)\,\frac{u^{17}+119u^{12}+187u^7+17u^2}{17u^{15}-187u^{10}+119u^5-1}.$$

The derivatives of w_1, w_2, w_3 contain as factors the icosahedral forms $u^{10}+11u^5-1$, $(u^{10}+11u^5-1)^2$, and $u^{20}-228u^{15}+494u^{10}+228u^5+1$. It is easy to see from the geometry of the configuration that this must be so, and this gives a check on the working.

(iv) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $w(\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$; **(iv *a*)** $u(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $w(\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$.

The fundamental functions associated with the u-triangles were obtained by Love,* who integrated the corresponding Schwarz-Christoffel equations. He gave the mapping of the triangle of angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{6}\pi$ upon a Z-half-plane in the form

$$Z = \frac{1-\wp^3(u)}{1+\wp^3(u)},$$

where $$\wp'^2(u) = 4\{\wp^3(u)-1\}.$$

These special elliptic functions have a real half-period ω_1 and two complex half-periods ω_2, ω_3, where

$$\omega_1 = \frac{1}{6}\,\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{3})}, \qquad \omega_2 = \epsilon\omega_1, \qquad \omega_3 = \epsilon^2\omega_1,$$

ϵ denoting $\exp(\frac{2}{3}i\pi)$.

* A. E. H. Love, *American J. of Math.* 11 (1889), 158–71. My notation does not agree with that of Love.

The angular points A, B, C of the u-triangle are ω_1, $\frac{4}{3}\omega_1+\frac{2}{3}\omega_2$, 0, corresponding to the Z-points 0, 1, -1. As, however, we have it always in mind that the mapping of a triangle is associated with the hypergeometric equation, we transform the Z-points 0, 1, -1 into the z-points 0, 1, ∞, so that

$$z = \frac{2Z}{Z+1} = 1-\wp^3(u).$$

The w-triangle of angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{3}\pi$ is the unit triangle of the tetrahedral configuration and six repetitions build up a quadrant of the w-plane. Six corresponding repetitions of the u-triangle build up the rectangle whose vertices are 0, ω_1, $2\omega_1+2\omega_2$, $\omega_1+2\omega_2$.

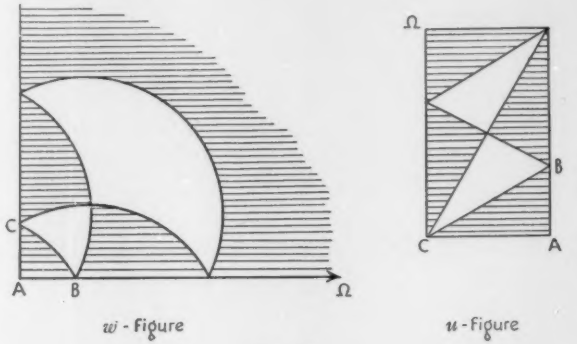


FIG. 4

The w-figure shows the configuration in Klein's second position of the tetrahedron relative to the axes of coordinates.

The quadrant is transformed into a half-plane by the relation $w^2 = \zeta$. With the conventional dimensions for the w-figure, C is the point $i(2-\sqrt{3})^{\frac{1}{2}}$, so that $\zeta = -(2-\sqrt{3})$.

Now consider the relation $Z = \wp(u;\Omega_1,\Omega_2)$ where Ω_1 is real, $\Omega_1+\Omega_2+\Omega_3 = 0$, $R(\Omega_3) = 0$, and $I(\Omega_3)$ is negative. This relation maps the rectangle whose vertices are 0, Ω_3, $\Omega_1+\Omega_3$, Ω_1 upon the half-plane of the variable Z. Hence, to map the rectangle whose vertices are $-\Omega_3$, 0, Ω_1, $\Omega_1-\Omega_3$, we have the relation

$$Z = \wp(u+\Omega_3;\Omega_1,\Omega_2),$$

i.e. a linear transformation of $\wp(u;\Omega_1,\Omega_2)$. To bring the points C, A, Ω into correspondence with three assigned points on the real axis

we must apply a further linear transformation to Z. Now, in our figure, $\Omega_1 = \omega_1$, $\Omega_3 = -(\omega_1+2\omega_2)$, and therefore $\Omega_2 = 2\omega_2$. Hence, writing $\wp(u; \omega_1, 2\omega_2)$ as $\wp_1(u)$, we have

$$\zeta = \frac{\alpha\wp_1(u)+\beta}{\gamma\wp_1(u)+\delta},$$

where α, β, γ, δ are to be determined.

We require that $\zeta = 0$ when $u = \omega_1$, ζ is infinite when $u = \omega_1+2\omega_2$ and $\zeta = -(2-\sqrt{3})$ when $u = 0$; i.e.

$$\zeta = -(2-\sqrt{3})\frac{\{\wp_1(u)-\wp_1(\omega_1)\}}{\{\wp_1(u)-\wp_1(\omega_1+2\omega_2)\}}.$$

But
$$\begin{aligned}\wp(u; \omega_1, 2\omega_2) &= \tfrac{1}{4}\wp(\tfrac{1}{2}u; \tfrac{1}{2}\omega_1, \omega_2)\\ &= \tfrac{1}{4}\{\wp(\tfrac{1}{2}u)+\wp(\tfrac{1}{2}u+\omega_1)-\wp(\omega_1)\},\end{aligned}$$

where the elliptic functions in the last expression are formed with periods $2\omega_1$, $2\omega_2$.

Hence
$$\zeta = -\frac{(2-\sqrt{3})\{\wp(\tfrac{1}{2}u)+\wp(\tfrac{1}{2}u+\omega_1)-2\wp(\tfrac{1}{2}\omega_1)\}}{\{\wp(\tfrac{1}{2}u)+\wp(\tfrac{1}{2}u+\omega_1)-2\wp(\tfrac{1}{2}\omega_1+\omega_2)\}}.$$

But $\wp(\frac{1}{2}\omega_1) = \sqrt{3}+1$, $\wp(\frac{1}{2}\omega_1+\omega_2) = -\sqrt{3}+1$, and hence

$$\zeta = -\frac{(\sqrt{3}-1)^2}{2}\frac{\{\wp(\tfrac{1}{2}u)-\sqrt{3}-1\}^2}{\{\wp(\tfrac{1}{2}u)+\sqrt{3}-1\}^2},$$

i.e.
$$w = \frac{i(\sqrt{3}-1)}{\sqrt{2}}\frac{\wp(\tfrac{1}{2}u)-\sqrt{3}-1}{\wp(\tfrac{1}{2}u)+\sqrt{3}-1},$$

the sign of w being chosen to make $I(w)$ positive, when u is a small positive number.

The tetrahedral triangle has angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{3}\pi$. We find two other functions admitting the same group by differentiation of the ground-forms $w_1^4 \pm 2\sqrt{3}w_1^2w_2^2 - w_2^4$. Our second and third functions may thus be

$$\frac{-\sqrt{3}w^2+1}{w^3+\sqrt{3}w}, \qquad \frac{\sqrt{3}w^2+1}{w^3-\sqrt{3}w}.$$

(v) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $w(\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$.

As in (ii) above, we may replace the dihedral group, $w(\frac{1}{2}, \frac{1}{2}, \frac{1}{3})$, by the anharmonic group.

Six repetitions of the unit triangle of the λ-configuration build up the λ-half-plane, and the corresponding repetitions of the u-triangle build up an equilateral triangle.

Love (l.c.) gives the representation of an equilateral triangle by means of the formula

$$Z = 2i/\wp'(u_1)$$

with the same special elliptic functions as in (iv) above. In his work the vertices of the equilateral triangle are 0, $\frac{4}{3}\omega_1+\frac{2}{3}\omega_2$, $\frac{2}{3}\omega_1+\frac{4}{3}\omega_2$ in the plane of u_1, and these correspond to the Z-points 0, 1, -1. In

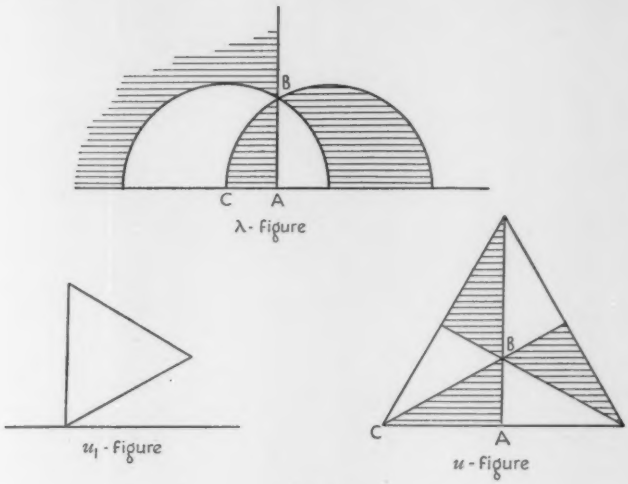


FIG. 5

our u-figure the vertices are 0, $2\omega_1$, $2\omega_1+2\omega_2$, and the corresponding λ-points are 0, 1, ∞. Hence we write

$$\lambda = \frac{2Z}{Z+1} \text{ and } u = \sqrt{3}u_1 \exp(-\tfrac{1}{6}i\pi) = i\sqrt{3}u_1\epsilon^2,$$

i.e. $\lambda = \dfrac{4i}{2i+\wp'(-iu\epsilon/\sqrt{3})} = \dfrac{4i}{2i-\wp'(iu/\sqrt{3})}$, for $\wp'(\epsilon u) = \wp'(u)$.

We take the same second and third functions as in (ii) above.

(vi) $u(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{2}, \frac{1}{4})$; **(vi*a*)** $u(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

The fact has already been remarked that an isosceles right-angled triangle may be built up of two repetitions of a smaller isosceles right-angled triangle. For this reason the fundamental function of the triangle may be presented in an endless variety of forms, connected by transformations of the special elliptic functions. I find it convenient not to use the form given by Love (l.c.) but to obtain another.

It is found by means of the Schwarz-Christoffel formula that a square in the u-plane is mapped on a Z-half-plane by the formula

$$Z = \wp(u),$$

where

$$\wp'^2(u) = 4\{\wp^3(u) - \wp(u)\}.$$

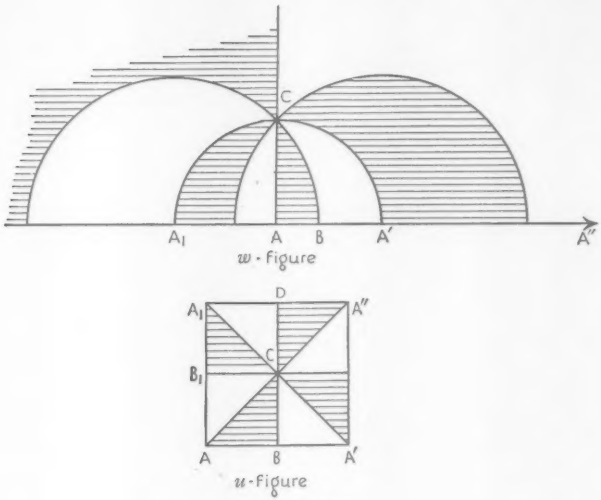


FIG. 6

These elliptic functions have a real half-period ω_1, and an imaginary half-period ω_3, where*

$$\omega_1 = \frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{4})} \text{ and } \omega_3 = -i\omega_1.$$

If now the u-point describes the contour of the square whose vertices are $-2\omega_3$, $-\omega_3$, $\omega_1-\omega_3$, $\omega_1-2\omega_3$, the Z-point describes the real axis from $-\infty$ to ∞, the points $-\infty$, -1, 0, 1 corresponding to the vertices A_1, B_1, C, D of the u-figure (Fig. 6).

By employing Burnside's method† for figures built up of repetitions of another figure, we find that the triangle A_1B_1C is mapped on a z-half-plane by the formula $z = 1/\wp^2(u)$, the z-points which

* I take $-i\omega_1$ in order to comply with the conventions $I(\omega_2/\omega_1)$ positive and $\omega_1+\omega_2+\omega_3 = 0$.

† W. Burnside, *Proc. London Math. Soc.* (1) 24 (1893), 187–206 (194–7).

correspond to the vertices being 0, 1, ∞. But the triangle ABC is a repetition of the triangle A_1B_1C. Hence $1/\wp^2(u)$ is the fundamental function for this triangle also.

Now eight repetitions of the w-triangle build up a half-plane, and the corresponding repetitions of the u-figure build up the square whose vertices A, A', A'', A_1 are 0, $2\omega_1$, $2\omega_1-2\omega_3$, $-2\omega_3$. In the w-figure A, A', A'', A_1 are 0, 1, ∞, -1. The relation between w, u is obviously a transformation of that between Z, u. In fact, we have merely to replace u by $\frac{1}{2}u-\omega_3$, and apply a linear transformation to Z to obtain w, so that

$$w = \frac{\alpha\wp(\frac{1}{2}u)+\beta}{\gamma\wp(\frac{1}{2}u)+\delta},$$

where α, β, γ, δ are to be determined.

Now $\alpha = 0$ since $w = 0$, when $u = 0$; $\delta = 0$ since w is infinite when $u = 2\omega_1+2\omega_2$; $\beta = \gamma$ since $w = 1$ when $u = 2\omega_1$,

i.e.
$$w = \frac{1}{\wp(\frac{1}{2}u)}.$$

Two other functions are obtained by the method of differentiation from the fundamental forms $w_1^4+w_2^4$, w_1w_2 associated with a triangle of angles $\frac{1}{2}\pi$, $\frac{1}{2}\pi$, $\frac{1}{4}\pi$. They are $-1/w^3$ and $-w$.

(vii) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $w(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$; **(vii *a*)** $u(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $w(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

We note first that, the groups of u, w being identical, u itself will suffice for one of the two functions we seek. The figure shows a generalized triangle which is associated with the same group.

The representation of this w-figure is given by the formula

$$\frac{dw}{dz} = K\frac{z^{\frac{1}{3}}}{(z-1)^{\frac{1}{3}}},$$

where K is a constant to be determined by the scale of the figure, the z-points 0, 1, ∞ corresponding to A, B, C. (This is not quite the ordinary Schwarz-Christoffel formula, but a slight modification which applies when one of the angular points of the rectilinear figure is at infinity.) Writing, from (iv) above, $z = 1-\wp^3(u)$, and adjusting the constant to convenience, we find that

$$w = \alpha\left\{6u-\frac{\wp'(u)}{\wp^2(u)}\right\}+\beta.$$

But when $u = 0$, $w = 0$, and therefore $\beta = 0$, and when $u = \omega_1$, $w = \omega_1$, and therefore $\alpha = \frac{1}{6}$.

The two functions we require are

$$u, \qquad u - \frac{1}{6}\frac{\wp'(u)}{\wp^2(u)}.$$

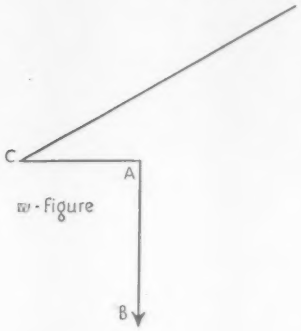


FIG. 7

(viii) $u(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}), w(\frac{1}{2}, \frac{2}{3}, \frac{1}{6})$; **(viii *a*)** $u(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), w(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$.

The figures (Fig. 8) show two generalized triangles associated with the required group. Their representations are given by

$$\frac{dw_1}{dz} = K_1 \frac{1}{z^{\frac{1}{3}}(z-1)^{\frac{1}{3}}}, \qquad \frac{dw_2}{dz} = K_2 \frac{z^{\frac{1}{3}}}{(z-1)^{\frac{1}{3}}}.$$

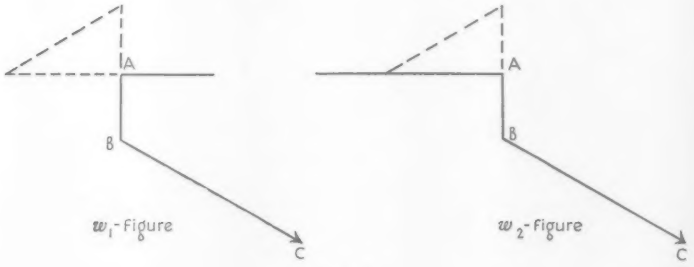


FIG. 8

Again, writing $z = 1 - \wp^2(u)$, we get

$$w_1 = \alpha\zeta(u) + \beta, \qquad w_2 = \gamma\{\zeta(u) + \tfrac{1}{6}\wp^2(u)\wp'(u)\} + \delta.$$

It is easy to see that the transformation $W = -w$ corresponds to

the transformation $U = -u$. Hence β, δ both vanish; also, when $u = \omega_1$, $w = \omega_1$; i.e.

$$w_1 = \frac{\omega_1}{\eta_1}\zeta(u), \qquad w_2 = \frac{\omega_1}{\eta_1}\{\zeta(u)+\tfrac{1}{6}\wp^2(u)\wp'(u)\}.$$

Since $\zeta(\epsilon u) = \epsilon^2\zeta(u)$, $\eta_2 = \epsilon^2\eta_1$, and hence, from the formula $\eta_1\omega_2 - \eta_2\omega_1 = \frac{1}{2}i\pi$, we find that $\eta_1\omega_1 = \pi/2\sqrt{3}$, so that the constant η_1 is known.

(ix) $u(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

As in (vii), u itself will supply one of the two functions we seek and the figure shows a generalized triangle associated with the same group.

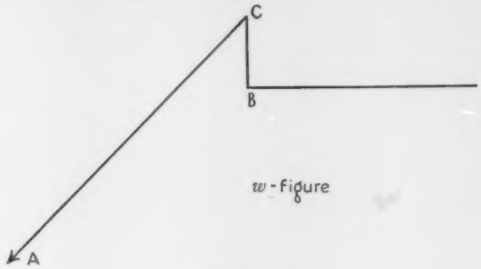


FIG. 9

The formula for its representation is

$$\frac{dw}{dz} = K\frac{(z-1)^{\frac{1}{2}}}{z^{\frac{1}{4}}}.$$

Writing, from (vi) above, $z = 1/\wp^2(u)$, we find that

$$w = \alpha\{4u - \wp'(u)\} + \beta.$$

Since the transformation $W = -w$ corresponds to the transformation $U = -u$, then $\beta = 0$; since $w = \omega_1$ when $u = \omega_1$, $\alpha = \frac{1}{4}$.

The two functions required are

$$u, \qquad u - \tfrac{1}{4}\wp'(u).$$

(x) $u(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$, $w(\frac{1}{2}, \frac{1}{4}, \frac{3}{4})$.

The figure shows two generalized triangles associated with the required group. Their representations are given by

$$\frac{dw_1}{dz} = \frac{K_1}{z^{\frac{1}{4}}(z-1)^{\frac{1}{2}}}, \qquad \frac{dw_2}{dz} = \frac{K_2(z-1)^{\frac{1}{2}}}{z^{\frac{1}{4}}}.$$

Writing $z = 1/\wp^2(u)$ again, we find that

$$w_1 = \alpha\zeta(u)+\beta, \qquad w_2 = \gamma\{\zeta(u)+\tfrac{1}{2}\wp(u)\wp'(u)\}+\delta.$$

Since the transformation $W = -w$ corresponds to the transformation $U = -u$, β, δ both vanish; since $w = \omega_1$ when $u = \omega_1$, $\alpha = \gamma = \omega_1/\eta_1$.

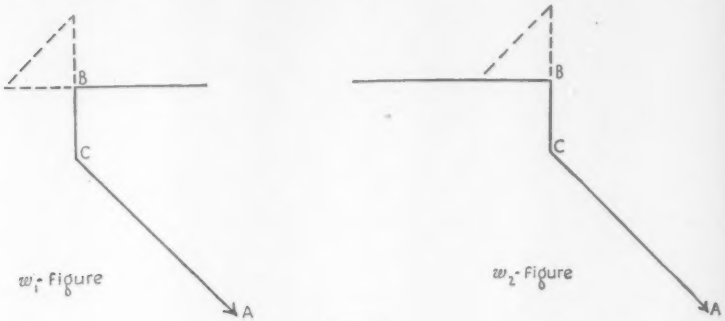


FIG. 10

The two functions required are

$$\frac{\omega_1}{\eta_1}\zeta(u), \qquad \frac{\omega_1}{\eta_1}\{\zeta(u)+\tfrac{1}{2}\wp(u)\wp'(u)\}.$$

Now for these elliptic functions $\zeta(-iu) = i\zeta(u)$, and therefore $\eta_3 = i\eta_1$, and hence, from the formula $\eta_3\omega_1-\eta_1\omega_3 = \frac{1}{2}i\pi$, we find that $\eta_1\omega_1 = \frac{1}{4}\pi$, so that the constant η_1 is known.

We note that the general formulae in §§ (vii)—(x) take specially simple forms. They are, after obvious simplifications,

(vii) $w = u+\wp(u)\wp'(u)\,F\{\wp^3(u)\}$.

(viii) $w = \dfrac{\omega_1}{\eta_1}\zeta(u)+\wp^2(u)\wp'(u)\,F\{\wp^3(u)\}$.

(ix) $w = u+\wp'(u)\,F\{\wp^2(u)\}$.

(x) $w = \dfrac{\omega_1}{\eta_1}\zeta(u)+\wp(u)\wp'(u)\,F\{\wp^2(u)\}$.

THE REPRESENTATION OF A NUMBER AS A SUM OF FIVE OR MORE SQUARES

By E. MAITLAND WRIGHT (*Oxford*)

[Received 5 July 1932]

Introduction

1. THE problem considered here is that of representing a number n as a sum of squares 'almost proportional' to assigned positive numbers. If

$$0 < \beta < 1,$$

and if

$$\lambda_1, \lambda_2, \ldots, \lambda_s$$

are any assigned positive numbers whose sum is unity, is it possible to express every positive integer n in the form

$$n = m_1^2 + m_2^2 + \ldots + m_s^2, \tag{1.01}$$

where

$$|m_i^2 - \lambda_i n| = O(n^{1-\beta}), \tag{1.02}$$

and what is the largest value of β for which this is true? In this paper I shall show that such a representation is always possible provided that

$$s \geqslant 5, \qquad \beta < \alpha = \alpha(s) = \min\left\{\frac{s-4}{3s-4}, \frac{s-2}{2(2s-1)}\right\}. \tag{1.03}$$

On the other hand, such a representation is not possible for all n if $\beta > \frac{1}{2}$. Let $[v]$ denote the greatest integer less than or equal to v. Let us take r any positive integer and n such that

$$[\lambda_i n] = r^2 + r.$$

Then,

$$\lambda_i n \geqslant 2, \qquad [\lambda_i n] > \tfrac{1}{2}\lambda_i n, \qquad r^2 \geqslant \tfrac{1}{2}(r^2+r) = \tfrac{1}{2}[\lambda_i n] > \tfrac{1}{4}\lambda_i n.$$

Also, if m_i^2 is the nearest square to $\lambda_i n$,

$$|m_i^2 - \lambda_i n| \geqslant r \geqslant \tfrac{1}{2}\lambda_i^{\frac{1}{2}} n^{\frac{1}{2}}.$$

If we let r run through all the positive integers, we obtain an infinite sequence of values of n for which the last result is true. But (1.02) cannot then be true.

The question remains unanswered if β lies in the closed interval

$$\alpha(s) \leqslant \beta \leqslant \tfrac{1}{2}.$$

Since $\alpha(s) < \frac{1}{4}$, this gap is considerable.

My method is based on the Vinogradoff method* for the solution

* M. Gelbke, *Math. Annalen*, 105 (1931), 637–52, gives this method in its latest and simplest form.

of Waring's problem. Incidentally, I find an asymptotic formula for the number of solutions of (1.01) satisfying the conditions

$$\lambda_i n - A_i' n^{1-\beta} < m_i^2 \leqslant \lambda_i n + A_i n^{1-\beta} \qquad (i = 1, 2, \ldots, s),$$

where $A_1, A_2, \ldots, A_s, \quad A_1', A_2', \ldots, A_s'$

are any assigned positive numbers.

I hope later to discuss the same problem for powers higher than the second, and for 3 and 4 squares. In the latter case, certain less complete results are obtainable.

Notation

2. In what follows s, i, n, h, j are positive integers, $1 \leqslant i \leqslant s$, and we write $\prod$ for $\prod_{i=1}^{s}$. Also

$$\beta, \lambda_i, A_i, A_i' \qquad (i = 1, 2, \ldots, s)$$

are any assigned positive numbers subject only to the conditions that

$$0 < \beta < 1, \qquad \lambda_1 + \lambda_2 + \ldots + \lambda_s = 1.$$

Later a more restricted condition will be imposed on β. We write

$$\alpha = \alpha(s) = \min\left\{\frac{s-4}{3s-4}, \frac{s-2}{2(2s-1)}\right\} = \begin{cases} \dfrac{s-4}{3s-4} & (5 \leqslant s \leqslant 8), \\ \dfrac{s-2}{2(2s-1)} & (s \geqslant 8), \end{cases}$$

$$\gamma = \gamma(s, \beta) = \tfrac{1}{2}s - 1 - (s-1)\beta,$$

$$A = \sum_{i=1}^{s} A_i, \qquad A' = \sum_{i=1}^{s} A_i',$$

$$\tau(z) = z^{s-1} \quad (z > 0), \qquad \tau(z) = 0 \quad (z \leqslant 0),$$

$$T = T(A_1, \ldots, A_s, A_1', \ldots, A_s')$$

$$= \tau(A') - \sum_{1 \leqslant i \leqslant s} \tau(A' - A_i - A_i') +$$

$$+ \sum_{1 \leqslant i < i' \leqslant s} \tau(A' - A_i - A_i' - A_{i'} - A_{i'}') - \sum_{1 \leqslant i < i' < i'' \leqslant s} \ldots + (-1)^s \tau(-A).$$

We use ϵ to denote a positive number, to be thought of as small; its choice will be subsequent to that of s and β. C and c are positive numbers, whose values vary from one occurrence to another. C depends at most on

$$s, \beta, \epsilon, \lambda_1, \ldots, \lambda_s, A_1, \ldots, A_s, A_1', \ldots, A_s';$$

c depends on s and β only. The constant implied in the $O(\ )$ notation is of the type C.

The relation

$$(1-x)^{-s} = \sum_{m=0}^{\infty} d_m x^m = \sum_{m=-\infty}^{+\infty} d_m x^m \qquad (|x| < 1),$$

defines $d_m = d(m,s)$, so that d_m is zero if m is negative.

We write

$$\rho = \exp(2\pi i p/q) \quad ((p,q)=1), \qquad S_\rho = \sum_{h=1}^{q} \rho^{h^s},$$

$$A_s(q,j) = q^{-s} \sum_{\rho} S_\rho^s \rho^{-j},$$

where ρ runs through all primitive qth roots of unity;

$$\mathfrak{S}(j,\lambda) = \sum_{1 \leqslant q \leqslant \lambda} A_s(q,j),$$

$$\mathfrak{S}(j) = \sum_{q=1}^{\infty} A_s(q,j).$$

$$N_i = [\lambda_i n + A_i n^{1-\beta}], \qquad N_i' = [\lambda_i n - A_i' n^{1-\beta} + 1],$$

$$L = n - \sum_{i=1}^{s} N_i' = A' n^{1-\beta} + O(1),$$

$$M_i = N_i - N_i' + 1 = (A_i + A_i') n^{1-\beta} + O(1).$$

We define $r(n)$ as the number of solutions of

$$n = m_1^2 + m_2^2 + \ldots + m_s^2$$

such that

$$m_i \geqslant 0, \qquad \lambda_i n - A_i' n^{1-\beta} < m_i^2 \leqslant \lambda_i n + A_i n^{1-\beta} \quad (i = 1, 2, \ldots, s),$$

or, what is the same thing,

$$N_i' \leqslant m_i^2 \leqslant N_i \qquad (i = 1, 2, \ldots, s);$$

$\nu(n)$ is the number of solutions of the equation

$$n = j_1 + j_2 + \ldots + j_s, \tag{2.01}$$

which satisfy the conditions

$$N_i' \leqslant j_i \leqslant N_i \qquad (i = 1, 2, \ldots, s). \tag{2.02}$$

We write

$$f_i(x) = \sum_{N_i' \leqslant h^s \leqslant N_i} x^{h^s},$$

$$\psi_{\rho,i}(x) = \frac{S_\rho}{2q} \sum_{j=N_i'}^{N_i} j^{-\frac{1}{2}} \left(\frac{x}{\rho}\right)^j.$$

Then $r(n)$ and $\nu(n)$ are the coefficients of x^n in

$$\prod f_i(x), \qquad \prod \left\{ \sum_{j=N_i'}^{N_i} x^j \right\}$$

respectively. We define $\mu(n)$ as the coefficient of x^n in

$$\prod \{\psi_{1,i}(x)\}.$$

Then the coefficient of x^n in

$$\prod \{\psi_{\rho,i}(x)\}$$

is $$q^{-s}S_\rho^s \rho^{-n}\mu(n).$$

We take the Farey dissection of order

$$l = [n^{\frac{1}{2}(1-\beta)}]$$

on the circle $\mathfrak{K}$ ($|x| = 1$). Then associated with every point ρ on $\mathfrak{K}$ for which

$$q \leqslant n^{\frac{1}{2}(1-\beta)}$$

we have an arc $\mathfrak{M}$. For each ρ we write

$$x = \rho e^{2\pi i y}, \qquad |y| \leqslant \tfrac{1}{2}.$$

Then we know that $$|y| < (ql)^{-1}$$

on the arc $\mathfrak{M}$ associated with ρ; while every point for which

$$|y| \leqslant (2ql)^{-1}$$

certainly lies on this $\mathfrak{M}$. Each arc $\mathfrak{M}$ is divided as follows. The part for which

$$|y| \leqslant q^{-\frac{3}{2}} n^{-\frac{1}{2}(1-\beta)}$$

is denoted by $\mathfrak{M}'$, the remainder, if any, by $\mathfrak{M}''$.

The values of $\mu(n)$ and $\nu(n)$

3.1. We first find the asymptotic value of $\nu(n)$, and from this deduce that of $\mu(n)$.

LEMMA 1. *If $s \geqslant 2$, then*

$$d_m = \{\tau(m)/\Gamma(s)\} + O(m^{s-2}) \qquad (m \geqslant 1),$$
$$d_0 = 1,$$
$$d_m = \tau(m)/\Gamma(s) \qquad (m < 0).$$

If $m \geqslant 1$,

$$d_m = \frac{s(1+s)\ldots(m+s-1)}{1.2\ldots m} = \frac{(m+1)\ldots(m+s-1)}{1.2\ldots(s-1)}$$
$$= \{m^{s-1}/\Gamma(s)\} + O(m^{s-2}) = \{\tau(m)/\Gamma(s)\} + O(m^{s-2}).$$

If $m < 0$, $$d_m = 0 = \tau(m)/\Gamma(s).$$

3.2. LEMMA 2. *If $s \geqslant 2$, then*

$$\left|\nu(n) - \frac{Tn^{(s-1)(1-\beta)}}{\Gamma(s)}\right| < Cn^{(s-2)(1-\beta)}.$$

We see that $\nu(n)$ is the coefficient of x^n in

$$\prod \left\{\sum_{j=N_i}^{N_i'} x^j\right\} = x^{n-L} \prod \left\{\sum_{j=0}^{M_i-1} x^j\right\} = \frac{x^{n-L}\prod (1-x^{M_i})}{(1-x)^s},$$

and is therefore the coefficient of x^L in

$$\frac{\prod(1-x^{M_i})}{(1-x)^s} = \frac{1-x^{M_1}-x^{M_2}-\ldots+x^{M_1+M_2}+\ldots}{(1-x)^s}.$$

But this is obviously

$$d_L - \sum_{1\leqslant i\leqslant s} d_{L-M_i} + \sum_{1\leqslant i<i'\leqslant s} d_{L-M_i-M_{i'}} - \ldots$$

$$= \frac{1}{\Gamma(s)}\left\{\tau(L) - \sum_{1\leqslant i\leqslant s}\tau(L-M_i) + \sum_{1\leqslant i<i'\leqslant s}\tau(L-M_i-M_{i'}) - \ldots\right\} + O(L^{s-2}).$$

Then, if we substitute for L, M_i, we have

$$\nu(n) = \frac{n^{(s-1)(1-\beta)}}{\Gamma(s)}\left\{\tau(A') - \sum_{1\leqslant i\leqslant s}\tau(A'-A_i-A_{i'}) + \ldots\right\} + O(n^{(s-2)(1-\beta)})$$

$$= \frac{Tn^{(s-1)(1-\beta)}}{\Gamma(s)} + O(n^{(s-2)(1-\beta)}).$$

3.3. Lemma 3. *If $s \geqslant 2$, then $T > C$.*

We see from the definition that T is independent of n. Then, by Lemma 2, it is sufficient to show that

$$\nu(n) > Cn^{(s-1)(1-\beta)}$$

for all large n. Now $\nu(n)$ is the number of solutions of

$$L = h_1+h_2+\ldots+h_s \tag{3.31}$$

which satisfy the conditions

$$0 \leqslant h_i \leqslant M_i-1 \qquad (i = 1, 2, \ldots, s). \tag{3.32}$$

Let B be a positive number such that

$$sB < \min(A_1, A_2, \ldots, A_s, A_1', A_2', \ldots, A_s').$$

Let us choose h_i $(1 \leqslant i \leqslant s-1)$ so that

$$(A_i'-B)n^{1-\beta} < h_i \leqslant (A_i'+B)n^{1-\beta}. \tag{3.33}$$

We can choose h_i in $[2Bn^{1-\beta}]$ ways so as to satisfy this condition. Hence, there are $[2Bn^{1-\beta}]^{s-1}$ sets of numbers

$$h_1, h_2, \ldots, h_{s-1}$$

which satisfy (3.33). But these numbers also satisfy (3.32), since

$$0 < (A_i'-B)n^{1-\beta} < (A_i'+B)n^{1-\beta} < M_i-1,$$

if $n > C$. Also, we can take

$$h_s = L-h_1-h_2-\ldots-h_{s-1},$$

and we shall have, for sufficiently large n,

$$h_s \geqslant \{A_s'-(s-1)B\}n^{1-\beta}+O(1) > 0,$$

$$h_s < \{A_s'+(s-1)B\}n^{1-\beta}+O(1) < M_s-1.$$

Then to each of the sets satisfying (3.33) corresponds at least one solution of (3.31) satisfying (3.32). That is, for all sufficiently large n,

$$\nu(n) \geqslant [2Bn^{1-\beta}]^{s-1} > Cn^{(s-1)(1-\beta)}.$$

3.4. LEMMA 4. *If* $s \geqslant 2$,

$$\left|\mu(n) - \frac{T}{2^s\Gamma(s)(\prod \lambda_i)^{\frac{1}{2}}} n^{\gamma}\right| < Cn^{\gamma-\beta} + Cn^{\gamma-1+\beta}.$$

We have

$$\mu(n) = 2^{-s} \sum (\prod j_i^{-\frac{1}{2}}), \qquad \nu(n) = \sum 1,$$

the summations extending over all sets of integers $j_1, j_2, \ldots, j_s$ satisfying (2.01) and (2.02). For such values of j_i,

$$\prod N_i' \leqslant \prod j_i \leqslant \prod N_i.$$

Now

$$\prod N_i' = \prod [\lambda_i n - A_i' n^{1-\beta} + 1] = n^s(\prod \lambda_i)\{1 + O(n^{-\beta})\},$$

$$\prod N_i = \prod [\lambda_i n + A_i n^{1-\beta}] = n^s(\prod \lambda_i)\{1 + O(n^{-\beta})\}.$$

Hence,

$$\prod j_i^{-\frac{1}{2}} = n^{-\frac{1}{2}s}(\prod \lambda_i)^{-\frac{1}{2}}\{1 + O(n^{-\beta})\},$$

and so

$$\mu(n) = \frac{\nu(n)}{2^s(\prod \lambda_i)^{\frac{1}{2}} n^{\frac{1}{2}s}}\{1 + O(n^{-\beta})\}$$

$$= \frac{Tn^{\gamma}}{2^s\Gamma(s)(\prod \lambda_i)^{\frac{1}{2}}}\{1 + O(n^{-\beta}) + O(n^{-(1-\beta)})\},$$

by Lemma 2.

We see from this lemma that

$$\mu(n) < Cn^{\gamma}.$$

Known results

4.1. The results in this section are not new, but for convenience we give them here in the form in which we shall require them.

LEMMA 5. *If* m *and* r *are integers,* m *being positive, then*

$$\left|\sum_{h=r+1}^{r+m} \rho^{h^2}\right| < Cq^{\epsilon}m^{\epsilon}(mq^{-\frac{1}{2}} + q^{\frac{1}{2}}).$$

It is known that*

$$\left|\sum_{h=r+1}^{r+m} \rho^{h^2}\right|^2 < Cq^{\epsilon}m^{\epsilon}(m + m^2q^{-1} + q) < Cq^{\epsilon}m^{\epsilon}(mq^{-\frac{1}{2}} + q^{\frac{1}{2}})^2,$$

and the result follows at once.

LEMMA 6. *We have* $\qquad |S_{\rho}| < Cq^{\frac{1}{2}+\epsilon}.$

This is Lemma 5 with $r = 0$, $m = q$.

* Landau, *Math. Zeitschrift*, 12 (1922), 226, Hilfsatz 2.

4.2. LEMMA 7. *If $s \geqslant 5$, then the series for $\mathfrak{S}(n)$ is convergent,*

$$|\mathfrak{S}(n)| < C,$$

and

$$|\mathfrak{S}(n) - \mathfrak{S}(n, n^{\frac{1}{2}(1-\beta)})| < Cn^{-\frac{1}{2}(1-\beta)}.$$

In Lemma 6, take $\epsilon < \frac{1}{4} - s^{-1}$.

Then we have

$$|A_s(q,n)| \leqslant q^{-s} \sum_{\rho} |S_\rho|^s \leqslant Cq^{1-\frac{1}{2}s+s\epsilon} < Cq^{-\frac{1}{2}s}.$$

Hence

$$\sum_{q>\lambda} |A_s(q,n)| \leqslant C \sum_{q>\lambda} q^{-\frac{1}{2}s} < C\lambda^{1-\frac{1}{2}s}.$$

This shows that the series for $\mathfrak{S}(n)$ is convergent. If we put $\lambda = \frac{1}{2}$, we have

$$|\mathfrak{S}(n)| < C;$$

and if we put

$$\lambda = n^{\frac{1}{2}(1-\beta)},$$

we have

$$|\mathfrak{S}(n) - \mathfrak{S}(n, n^{\frac{1}{2}(1-\beta)})| < Cn^{\frac{1}{2}(1-\frac{1}{2}s)(1-\beta)} \leqslant Cn^{-\frac{1}{2}(1-\beta)}.$$

4.3. LEMMA 8. *If $s \geqslant 5$ and n is sufficiently large, then*

$$\mathfrak{S}(n) > C. \tag{4.31}$$

If $s \geqslant 5$, the number of representations of n as the sum of s squares is* $\{\Gamma(\frac{1}{2}s)\}^{-1}\pi^{\frac{1}{2}s}n^{\frac{1}{2}s-1}\mathfrak{S}(n) + O(n^{\frac{1}{2}s-1-c})$, (4.32)

where $c = c(s) > 0$. Again, if $s = 5$, we have†

$$\mathfrak{S}(n) > C \sum \left(\frac{n}{m}\right)\frac{1}{m^2},$$

where m runs through all odd numbers prime to n and $\left(\frac{n}{m}\right)$ is the Legendre symbol. Since

$$\left(\frac{n}{1}\right) = 1, \qquad \left(\frac{n}{m}\right) = \pm 1,$$

we have, when $s = 5$,

$$\mathfrak{S}(n) > C\left(1 - \sum_{t=1}^{\infty} \frac{1}{(2t+1)^2}\right) = C(2 - \tfrac{1}{8}\pi^2) > C.$$

* No proof of this for general s appears to have been published, but one can readily be constructed by the Hardy-Littlewood method for the solution of Waring's problem or by the Vinogradoff method. Hardy (*Trans. American Math. Soc.*, 21 (1920), 255–84) has proved that the number of representations of n as the sum of s squares is exactly

$$\{\Gamma(\tfrac{1}{2}s)\}^{-1}\pi^{\frac{1}{2}s}n^{\frac{1}{2}s-1}\mathfrak{S}(n),$$

if $3 \leqslant s \leqslant 8$.

† Hardy, loc. cit.

Ostrowski* has shown that, if (4.31) is true for $s = 5$ and (4.32) is true for $s \geqslant 5$, then (4.31) is true for $s \geqslant 5$. This completes the proof of the lemma.

4.4. LEMMA 9. *We have*

$$\Big|\sum_{m=1}^{j} m^{-\frac{1}{2}} - 2j^{\frac{1}{2}}\Big| < C.$$

When $m \geqslant 1$,

$$2m^{\frac{1}{2}} - 2(m-1)^{\frac{1}{2}} = m^{-\frac{1}{2}} + O(m^{-\frac{3}{2}}),$$

and, by summation,

$$2j^{\frac{1}{2}} = \sum_{m=1}^{j} m^{-\frac{1}{2}} + \sum_{m=1}^{j} O(m^{-\frac{3}{2}}) = \sum_{m=1}^{j} m^{-\frac{1}{2}} + O(1).$$

Behaviour of $f_i(x)$ and $\psi_{\rho,i}(x)$ on $\mathfrak{R}$

5.1. LEMMA 10. *We have*

$$|\psi_{\rho,i}(x)| < Cn^{-\frac{1}{2}}q^{-\frac{1}{2}+\epsilon}\min(n^{1-\beta}, |y|^{-1}).$$

We have

$$N_i' = [\lambda_i n - A_i' n^{1-\beta} + 1] = \lambda_i n + O(n^{1-\beta}) > Cn,$$

if n is greater than a certain n_0, where n_0 is a number of the type C. Hence, in this case,

$$\begin{aligned} |\psi_{\rho,i}(x)| &\leqslant \frac{|S_\rho|}{2q}\Big(\sum_{j=N_i'}^{N_i} j^{-\frac{1}{2}}\Big) \\ &\leqslant Cq^{-\frac{1}{2}+\epsilon}N_i'^{-\frac{1}{2}}(N_i - N_i') \qquad \leqslant Cq^{-\frac{1}{2}+\epsilon}n^{\frac{1}{2}-\beta}. \end{aligned}$$

Again, if we write

$$w(j) = \sum_{m=N_i'}^{j} e^{2\pi i m y} \qquad (j \geqslant N_i'),$$

$$w(N_i' - 1) = 0,$$

then

$$|w(j)| \leqslant \frac{2}{|1 - e^{2\pi i y}|} < \frac{C}{|y|},$$

and

$$\begin{aligned} \sum_{j=N_i}^{N_i-1} j^{-\frac{1}{2}}\left(\frac{x}{\rho}\right)^j &= \sum_{j=N_i}^{N_i} j^{-\frac{1}{2}}\{w(j) - w(j-1)\} \\ &= \sum_{j=N_i}^{N_i-1}\{j^{-\frac{1}{2}} - (j+1)^{-\frac{1}{2}}\}w(j) + N_i^{-\frac{1}{2}}w(N_i). \end{aligned}$$

Hence,

$$\begin{aligned} |\psi_{\rho,i}(x)| &\leqslant \frac{C|S_\rho|}{q|y|}\left\{\sum_{j=N_i'}^{N_i-1} |j^{-\frac{1}{2}} - (j+1)^{-\frac{1}{2}}| + N_i^{-\frac{1}{2}}\right\} \\ &< Cq^{-\frac{1}{2}+\epsilon}|y|^{-1}(N_i')^{-\frac{1}{2}} < Cq^{-\frac{1}{2}+\epsilon}n^{-\frac{1}{2}}|y|^{-1}. \end{aligned}$$

* *Math. Zeitschrift*, 9 (1921), 28–34.

On the other hand, if $n \leqslant n_0$, then

$$q \leqslant n_0 = C, \qquad |y| \leqslant \tfrac{1}{2} = C,$$
$$|\psi_{\rho,i}(x)| \leqslant \tfrac{1}{2}(N_i - N_i' + 1) < Cn \leqslant Cn_0 = C,$$

and the lemma is obviously true.

5.2. LEMMA 11. *We have*

$$|f_i(x) - \psi_{\rho,i}(x)| < Cq^{\frac{1}{2}+\epsilon} \max(1, n^{1-\beta}|y|).$$

Let us write

$$u(j) = \sum_{1 \leqslant h^2 \leqslant j} \rho^{h^2}, \qquad v(j) = \tfrac{1}{2} q^{-1} S_\rho \sum_{1 \leqslant m \leqslant j} m^{-\frac{1}{2}}.$$

Then

$$u(j) = \sum_{1 \leqslant h \leqslant q[j^{\frac{1}{2}}/q]} \rho^{h^2} + \sum_{q[j^{\frac{1}{2}}/q] < h \leqslant [j^{\frac{1}{2}}]} \rho^{h^2}.$$

The second sum has less than q terms and so, by Lemma 5, is less in absolute value than $Cq^{\frac{1}{2}+\epsilon}$. The value of the first sum is obviously

$$S_\rho[j^{\frac{1}{2}}/q].$$

Hence

$$|u(j) - [j^{\frac{1}{2}}/q] S_\rho| < Cq^{\frac{1}{2}+\epsilon}.$$

Also,

$$|[j^{\frac{1}{2}}/q] S_\rho - (j^{\frac{1}{2}}/q) S_\rho| < |S_\rho| < Cq^{\frac{1}{2}+\epsilon}.$$

By Lemma 9,

$$|v(j) - (j^{\frac{1}{2}}/q) S_\rho| < C|S_\rho/q| < C.$$

Hence,

$$|u(j) - v(j)| < Cq^{\frac{1}{2}+\epsilon}.$$

Now we have

$$f_i(x) = \sum_{N_i' \leqslant h^2 \leqslant N_i} x^{h^2} = \sum_{N_i' \leqslant j \leqslant N_i} \{u(j) - u(j-1)\} \left(\frac{x}{\rho}\right)^j$$
$$= \left(1 - \frac{x}{\rho}\right) \sum_{j=N_i'}^{N_i - 1} u(j) \left(\frac{x}{\rho}\right)^j - u(N_i' - 1) \left(\frac{x}{\rho}\right)^{N_i'} + u(N_i) \left(\frac{x}{\rho}\right)^{N_i},$$

and similarly

$$\psi_{\rho,i}(x) = \left(1 - \frac{x}{\rho}\right) \sum_{j=N_i'}^{N_i - 1} v(j) \left(\frac{x}{\rho}\right)^j - v(N_i' - 1) \left(\frac{x}{\rho}\right)^{N_i'} + v(N_i) \left(\frac{x}{\rho}\right)^{N_i}.$$

Then, finally, we have

$$|f_i(x) - \psi_{\rho,i}(x)| \leqslant \left|1 - \frac{x}{\rho}\right| \sum_{j=N_i'}^{N_i - 1} |u(j) - v(j)| +$$
$$+ |u(N_i' - 1) - v(N_i' - 1)| + |u(N_i) - v(N_i)|$$
$$\leqslant Cq^{\frac{1}{2}+\epsilon} \left\{ \left|1 - \frac{x}{\rho}\right| (N_i - N_i') + 1 \right\} < Cq^{\frac{1}{2}+\epsilon}(n^{1-\beta}|y| + 1).$$

5.3. LEMMA 12. *If* $|y| \leqslant n^{\beta-1}$, *then*

$$\left|\prod f_i(x) - \prod \psi_{\rho,i}(x)\right| < Cq^{s\epsilon}(q^{\frac{1}{2}s} + n^{(\frac{1}{2}-\beta)(s-1)} q^{-\frac{1}{2}(s-2)}).$$

If $n^{\beta-1} \leqslant |y| \leqslant \frac{1}{2}$, *then*

$$\left|\prod f_i(x) - \prod \psi_{\rho,i}(x)\right| < Cq^{s\epsilon}(q^{\frac{1}{2}s} n^{s(1-\beta)} |y|^s + q^{-\frac{1}{2}s+1} n^{\frac{1}{2}(3-s)-\beta} |y|^{2-s}).$$

If $s = 1$, the lemma follows at once from Lemma 11. If $s > 1$, let us write

$$f_i = \psi_{\rho,i} + \Psi_{\rho,i}.$$

Then we have

$$\prod f_i - \prod \psi_{\rho,i} = \prod (\psi_{\rho,i} + \Psi_{\rho,i}) - \prod \psi_{\rho,i},$$

a sum of $2^s - 1$ terms. Each term consists of s factors, not more than $s-1$ of which are of the form $\psi_{\rho,i}$. By Lemmas 10 and 11, a term which contains r factors of the form $\psi_{\rho,i}$ and $s-r$ of the form $\Psi_{\rho,i}$ is less in absolute value than

$$Cn^{-\frac{1}{2}r} q^{\frac{1}{2}s - r + s\epsilon} \min(n^{r(1-\beta)}, |y|^{-r}) \max\{(n^{1-\beta}|y|)^{s-r}, 1\}. \tag{5.31}$$

When $|y| \leqslant n^{\beta-1}$, this is less than

$$Cn^{r(\frac{1}{2}-\beta)} q^{\frac{1}{2}s - r + s\epsilon},$$

and hence

$$\begin{aligned} \Big|\prod f_i(x) - \prod \psi_{\rho,i}(x)\Big| &< Cq^{\frac{1}{2}s+s\epsilon} \sum_{r=0}^{s-1} (n^{\frac{1}{2}-\beta} q^{-1})^r \\ &< Cq^{s\epsilon}(q^{\frac{1}{2}s} + n^{(\frac{1}{2}-\beta)(s-1)} q^{-\frac{1}{2}(s-2)}). \end{aligned}$$

When $|y| \geqslant n^{\beta-1}$, (5.31) is less than

$$C(q^{\frac{1}{2}+\epsilon}|y| n^{1-\beta})^s (q^{-1}|y|^{-2} n^{-\frac{3}{2}+\beta})^r,$$

and hence

$$\begin{aligned} \Big|\prod f_i(x) - \prod \psi_{\rho,i}(x)\Big| &< C(q^{\frac{1}{2}+\epsilon}|y| n^{1-\beta})^s \{1 + (q^{-1}|y|^{-2} n^{-\frac{3}{2}+\beta})^{s-1}\} \\ &< Cq^{s\epsilon}(q^{\frac{1}{2}s} n^{s(1-\beta)} |y|^s + q^{-\frac{1}{2}s+1} |y|^{2-s} n^{\frac{1}{2}(3-s)-\beta}). \end{aligned}$$

5.4. LEMMA 13. *On* $\mathfrak{M}''$,

$$|f_i(x)| < Cn^{\epsilon}(q|y|)^{-\frac{1}{2}-\epsilon}.$$

We write $m = n^{1-\beta}, \qquad x = \exp\big(2\pi i\{(p/q) + y\}\big),$

and we have

$$\frac{1}{q^{\frac{1}{2}} m^{\frac{1}{2}}} \leqslant |y| < \frac{1}{q[m^{\frac{1}{2}}]}, \tag{5.41}$$

$$q|y| < 2.$$

Then, by the theory of the Farey series, we know that there are two integers p_1, q_1 and a number y_1 such that

$$\begin{gathered} \frac{p_1}{q_1} + y_1 = \frac{p}{q} + y, \\ 0 \leqslant p_1 \leqslant q_1 \leqslant \frac{2}{q|y|}, \qquad (p_1, q_1) = 1, \\ |y_1| < \frac{q|y|}{2q_1} \leqslant \frac{1}{q_1^2}. \end{gathered} \tag{5.42}$$

First suppose that

$$\frac{p}{q} = \frac{p_1}{q_1}.$$

Then

$$p = p_1, \qquad q = q_1, \qquad y = y_1, \qquad |y_1| = \frac{q|y|}{q_1} > \frac{q|y|}{2q_1},$$

which contradicts (5.42). Hence,

$$\frac{p}{q} \neq \frac{p_1}{q_1}.$$

Again, suppose that $q_1 \leqslant [m^{\frac{1}{2}}]$.

Then there is an arc $\mathfrak{M}_1$ corresponding to p_1/q_1 in our original Farey dissection of the circumference $\mathfrak{K}$. Since, by (5.41) and (5.42),

$$|y_1| < \frac{q|y|}{2q_1} < \frac{1}{2q_1[m^{\frac{1}{2}}]},$$

x lies within this arc $\mathfrak{M}_1$ (that is, not at an end-point). But this is impossible, for x lies on the arc $\mathfrak{M}$. Hence

$$q_1 > [m^{\frac{1}{2}}],$$

and so

$$q_1 > m^{\frac{1}{2}} = n^{\frac{1}{2}(1-\beta)}.$$

We now write

$$\rho_1 = \exp(2\pi i p_1/q_1), \qquad u_1(j) = \sum_{N_i' \leqslant h^2 \leqslant j} \rho_1^{h^2} \quad (N_i' \leqslant j \leqslant N_i).$$

Then $u_1(j)$ contains at most

$$N_i^{\frac{1}{2}} - N_i'^{\frac{1}{2}} + 1 < Cn^{\frac{1}{2}-\beta}$$

terms, and we have

$$|u_1(j)| < Cq^{\epsilon} n^{\epsilon(\frac{1}{2}-\beta)} (n^{\frac{1}{2}-\beta} q_1^{-\frac{1}{2}} + q_1^{\frac{1}{2}}) < Cq_1^{\frac{1}{2}+\epsilon} n^{\epsilon(\frac{1}{2}-\beta)},$$

since

$$q_1 > n^{\frac{1}{2}(1-\beta)} > n^{\frac{1}{2}-\beta}.$$

Hence,

$$\begin{aligned} f_i(x) &= \sum_{N_i' \leqslant h^2 \leqslant N_i} x^{h^2} \\ &= \left(\frac{x}{\rho}\right)^{N_i'} u_1(N_i') + \sum_{j=N_i'+1}^{N_i} \left(\frac{x}{\rho_1}\right)^j \{u_1(j) - u_1(j-1)\} \\ &= \left(1 - \frac{x}{\rho_1}\right) \sum_{j=N_i'}^{N_i-1} \left(\frac{x}{\rho_1}\right)^j u_1(j) + \left(\frac{x}{\rho_1}\right)^{N_i} u_1(N_i). \end{aligned}$$

From this we have finally

$$\begin{aligned} |f_i(x)| &\leqslant \left|1 - \frac{x}{\rho_1}\right| \sum_{j=N_i'}^{N_i-1} |u_1(j)| + |u_1(N_i)| \\ &< Cn^{\epsilon} q_1^{\frac{1}{2}+\epsilon} \left(\left|1 - \frac{x}{\rho_1}\right| (N_i - N_{i'}) + 1 \right) \\ &< Cn^{\epsilon} q_1^{\frac{1}{2}+\epsilon} (n^{1-\beta}|y_1| + 1) < Cn^{\epsilon} q_1^{\frac{1}{2}+\epsilon} (n^{1-\beta} q_1^{-2} + 1) \\ &< Cn^{\epsilon} q_1^{\frac{1}{2}+\epsilon} < Cn^{\epsilon} (q|y|)^{-\frac{1}{2}-\epsilon}. \end{aligned}$$

Final lemmas

6.1. So far β has denoted any positive number less than unity. We now introduce the restriction

$$0 < \beta < \alpha,$$

and write $\delta = \alpha - \beta > 0.$

Finally, we choose ϵ so that

$$\epsilon < \delta/(2s+1).$$

LEMMA 14. *If* $s \geqslant 5$, $0 < \beta < \alpha$, *and*

$$E_1 = n^{\beta-1} \sum_{\mathfrak{M}'} q^{\frac{1}{2}s}, \qquad E_2 = n^{\frac{1}{2}(s-3)-\beta(s-2)} \sum_{\mathfrak{M}'} q^{-\frac{1}{2}(s-2)},$$

$$E_3 = n^{\frac{1}{2}(s-2)(1-\beta)} \sum_{\mathfrak{M}'} q^{-\frac{1}{2}(s+4)}, \qquad E_4 = n^{\frac{1}{2}(s-4)-\frac{1}{2}(s-1)\beta} \sum_{\mathfrak{M}'} q^{\frac{1}{2}(s-4)},$$

we have $E_t < Cn^{\gamma-2\delta+\epsilon} \qquad (t = 1, 2, 3, 4).$

There are at most q arcs $\mathfrak{M}'$ associated with a particular q. Hence, if ω is any real number depending on s and β only, we have, when $\omega > -2$,

$$\sum_{\mathfrak{M}'} q^{\omega} \leqslant \sum_{q \leqslant n^{\frac{1}{2}(1-\beta)}} q^{\omega+1} < Cn^{\frac{1}{2}(\omega+2)(1-\beta)},$$

and, when $\omega \leqslant -2$,

$$\sum_{\mathfrak{M}'} q^{\omega} \leqslant \sum_{q \leqslant n^{\frac{1}{2}(1-\beta)}} q^{\omega+1} \leqslant \sum_{q \leqslant n^{\frac{1}{2}(1-\beta)}} q^{-1} < C \log n < Cn^{\epsilon}.$$

Using these two results, we find that

$$\begin{aligned}
E_1 &< Cn^{\frac{1}{2}s(1-\beta)} && (s \geqslant 5),\\
E_2 &< Cn^{\frac{1}{2}(5-13\beta)} && (s = 5),\\
E_2 &< Cn^{\frac{1}{2}(s-3)-\beta(s-2)+\epsilon} && (s \geqslant 6),\\
E_3 &\leqslant Cn^{\frac{1}{2}s(1-\beta)} && (5 \leqslant s \leqslant 7),\\
E_3 &\leqslant Cn^{\frac{1}{2}(s-2)(1-\beta)+\epsilon} && (s \geqslant 8),\\
E_4 &\leqslant Cn^{\frac{1}{2}s(1-3\beta)} < Cn^{\frac{1}{2}s(1-\beta)} && (s \geqslant 5).
\end{aligned}$$

Hence it is sufficient to prove the inequalities

$$\begin{aligned}
&\tfrac{1}{4}s(1-\beta) \leqslant \gamma - 2\delta && (s \geqslant 5),\\
&\tfrac{1}{4}(5-13\beta) \leqslant \gamma - 2\delta && (s = 5),\\
&\tfrac{1}{2}(s-3)-\beta(s-2) \leqslant \gamma - 2\delta && (s \geqslant 6),\\
&\tfrac{1}{3}(s-2)(1-\beta) \leqslant \gamma - 2\delta && (s \geqslant 8).
\end{aligned}$$

To prove the first we have

$$\begin{aligned}
\gamma - \tfrac{1}{4}s(1-\beta) &= \tfrac{1}{4}\{(s-4)-\beta(3s-4)\}\\
&= \tfrac{1}{4}\{(s-4)-\alpha(3s-4)\} + \tfrac{1}{4}\delta(3s-4) \geqslant \tfrac{11}{4}\delta > 2\delta,
\end{aligned}$$

since, by definition, $(3s-4)\alpha \leqslant (s-4).$

For the second, when $s = 5$, we have

$$\gamma = \tfrac{3}{2} - 4\beta, \qquad \beta = \alpha - \delta, \qquad \delta < \alpha = \tfrac{1}{11},$$

$$\gamma - \tfrac{1}{4}(5-13\beta) = \tfrac{1}{4}(1-3\beta) = \tfrac{1}{4}(1-3\alpha) + \tfrac{3}{4}\delta = 2\alpha + \tfrac{3}{4}\delta > \tfrac{11}{4}\delta > 2\delta.$$

For the third,

$$\gamma - \tfrac{1}{2}(s-3) + \beta(s-2) = \tfrac{1}{2} - \beta = \tfrac{1}{2} - \alpha + \delta > \alpha + \delta > 2\delta,$$

since $\alpha < \frac{1}{4}$. For the fourth, when $s \geqslant 8$, we have

$$2\alpha(2s-1) = s-2,$$

$$\gamma - \tfrac{1}{3}(s-2)(1-\beta) = \tfrac{1}{6}\{(s-2) - 2\beta(2s-1)\} = \tfrac{1}{3}\delta(2s-1) \geqslant 5\delta > 2\delta.$$

6.2. LEMMA 15. *If $s \geqslant 5$ and $0 < \beta < \alpha$, then*

$$I_1 = \sum_{\mathfrak{M}'} \int_{\mathfrak{M}'} \left| \prod f_i(x) - \prod \psi_{\rho,i}(x) \right| |dx| < Cn^{\gamma-\delta}.$$

By Lemma 12, we have

$$\int_{\mathfrak{M}'} \left| \prod f_i(x) - \prod \psi_{\rho,i}(x) \right| |dx|$$

$$\leqslant Cq^{s\epsilon}\Big\{ (q^{\frac{1}{2}s} + n^{(\frac{1}{2}-\beta)(s-1)} q^{-\frac{1}{2}(s-2)}) \int_0^{n^{\beta-1}} dy +$$

$$+ q^{\frac{1}{2}s} n^{s(1-\beta)} \int_{n^{\beta-1}}^{q^{-\frac{1}{2}}n^{-\frac{1}{2}(1-\beta)}} y^s \, dy + q^{-\frac{1}{2}s+1} n^{\frac{1}{2}(3-s)-\beta} \int_{n^{\beta-1}}^{\infty} y^{2-s} \, dy \Big\}$$

$$\leqslant Cn^{s\epsilon}(n^{\beta-1}q^{\frac{1}{2}s} + n^{\frac{1}{2}(s-3)-\beta(s-2)} q^{-\frac{1}{2}(s-2)} + n^{\frac{1}{2}(s-2)(1-\beta)} q^{-\frac{1}{2}(s+4)}),$$

since $$q^{s\epsilon} < n^{s\epsilon}.$$

We now sum over all the arcs $\mathfrak{M}'$, and we find

$$I_1 < Cn^{s\epsilon}(E_1 + E_2 + E_3) < Cn^{\gamma-2\delta+(s+1)\epsilon} < Cn^{\gamma-\delta}.$$

6.3. LEMMA 16. *If $s \geqslant 5$ and $0 < \beta < \alpha$, then*

$$I_2 = \sum_{\mathfrak{M}''} \int_{\mathfrak{M}''} \left| \prod f_i(x) \right| |dx| < Cn^{\gamma-\delta}.$$

By Lemma 13, we have

$$\int_{\mathfrak{M}''} \left| \prod f_i(x) \right| |dx| < Cn^{s\epsilon} q^{-\frac{1}{2}s - s\epsilon} \int_{q^{-\frac{1}{2}}n^{-\frac{1}{2}(1-\beta)}} |y|^{-\frac{1}{2}s - s\epsilon} \, dy$$

$$= Cn^{s\epsilon} q^{-\frac{1}{2}s - s\epsilon} (q^{\frac{1}{2}} n^{\frac{1}{2}(1-\beta)})^{\frac{1}{2}s + s\epsilon - 1} < Cn^{2s\epsilon + \frac{1}{2}(1-\beta)(s-2)} q^{-\frac{1}{2}(s+4)}.$$

We see that every $\mathfrak{M}''$ is associated with an $\mathfrak{M}'$ with the same value of q. Hence,

$$I_2 < Cn^{2s\epsilon + \frac{1}{2}(1-\beta)(s-2)} \sum_{\mathfrak{M}'} q^{-\frac{1}{2}(s+4)}$$

$$= Cn^{2s\epsilon} E_3 < Cn^{\gamma - 2\delta + (2s+1)\epsilon} < Cn^{\gamma-\delta}.$$

6.4. LEMMA 17. *If $s \geqslant 5$ and $0 < \beta < \alpha$, then*

$$I_3 = \sum_{\mathfrak{M}'} \int_{\mathfrak{R}-\mathfrak{M}'} |\prod \psi_{\rho,i}(x)|\,|dx| < Cn^{\gamma-\delta}.$$

We have, by Lemma 10,

$$|\psi_{\rho,i}(x)| < Cn^{-\frac{1}{2}}q^{-\frac{1}{2}+\epsilon}|y|^{-1}.$$

Hence
$$\int_{\mathfrak{R}-\mathfrak{M}'} |\prod \psi_{\rho,i}(x)|\,|dx| < Cn^{-\frac{1}{2}s}q^{-\frac{1}{2}s+s\epsilon} \int_{q^{-\frac{1}{2}}n^{-\frac{1}{2}(1-\beta)}}^{\infty} y^{-s}\,dy$$
$$= Cq^{s\epsilon}n^{-\frac{1}{2}s}q^{-\frac{1}{2}s}(q^{\frac{1}{2}}n^{\frac{1}{2}(1-\beta)})^{s-1}$$
$$< Cn^{s\epsilon+\frac{1}{2}(s-4)-\frac{1}{2}(s-1)\beta}q^{\frac{1}{2}(s-4)};$$

and so
$$I_3 < Cn^{s\epsilon}E_4 < Cn^{\gamma-2\delta+(s+1)\epsilon} < Cn^{\gamma-\delta}.$$

Proof of Theorems

7.1. THEOREM 1. *If $s \geqslant 5$ and $0 < \beta < \alpha$, then*

$$r(n) = \frac{T}{2^s\Gamma(s)(\prod \lambda_i)^{\frac{1}{2}}} S(n)n^{\gamma} + O(n^{\gamma-c}),$$

where $c = c(s,\beta) > 0$.

The coefficient of x^n in

$$\sum_{\rho} \{\prod \psi_{\rho,i}(x)\},$$

where ρ runs through all primitive qth roots of unity, is

$$\mu(n) \sum_{\rho} q^{-s}S_{\rho}^{s}\rho^{-n} = \mu(n)A_s(q,n).$$

Hence, the coefficient of x^n in

$$\sum_{q \leqslant n^{\frac{1}{2}(1-\beta)}} \sum_{\rho} \{\prod \psi_{\rho,i}(x)\}$$

is
$$\mu(n)\mathfrak{S}(n, n^{\frac{1}{2}(1-\beta)}).$$

From this we have, by Cauchy's theorem,

$$r(n) - \mu(n)S(n, n^{\frac{1}{2}(1-\beta)}) = \frac{1}{2\pi i}\int_{\mathfrak{R}} \Big\{\prod f_i(x) - \sum_{q \leqslant n^{\frac{1}{2}(1-\beta)}} \sum_{\rho} \prod \psi_{\rho,i}(x)\Big\} \frac{dx}{x^{n+1}}$$

$$= \frac{1}{2\pi i}\Big(\sum_{\mathfrak{M}'} \int_{\mathfrak{M}'} \{\prod f_i(x) - \prod \psi_{\rho,i}(x)\} \frac{dx}{x^{n+1}} + \sum_{\mathfrak{M}''} \int_{\mathfrak{M}''} \{\prod f_i(x)\} \frac{dx}{x^{n+1}} -$$
$$- \sum_{\mathfrak{M}'} \int_{\mathfrak{R}-\mathfrak{M}} \{\prod \psi_{\rho,i}(x)\} \frac{dx}{x^{n+1}}\Big).$$

Then, by Lemmas 15, 16, and 17,

$$|r(n) - \mu(n)\mathfrak{S}(n, n^{\frac{1}{2}(1-\beta)})| \leqslant \frac{1}{2\pi}(I_1 + I_2 + I_3) < Cn^{\gamma-\delta}.$$

Finally, by Lemmas 4 and 7,

$$\begin{aligned} r(n) &= \mu(n)\mathfrak{S}(n, n^{\frac{1}{2}(1-\beta)}) + O(n^{\gamma-\delta}) \\ &= \mu(n)\mathfrak{S}(n) + O(n^{\gamma-\frac{1}{2}(1-\beta)}) + O(n^{\gamma-\delta}) \\ &= \frac{T n^{\gamma}\mathfrak{S}(n)}{2^s\Gamma(s)(\prod\lambda_i)^{\frac{1}{2}}} + G, \end{aligned}$$

where G is a number of order

$$O(n^{\gamma-\delta}) + O(n^{\gamma-\frac{1}{2}(1-\beta)}) + O(n^{\gamma-\beta}) + O(n^{\gamma-1+\beta}) = O(n^{\gamma-c}).$$

7.2. THEOREM 2. *If $s \geqslant 5$ and $0 < \beta < \alpha$, there is a number*

$$n_0 = n_0(s, \beta, \lambda_1, \ldots, \lambda_s, A_1, \ldots, A_s, A_1', \ldots, A_s')$$

such that, for all $n > n_0$, there is a solution of the equation

$$n = m_1^2 + m_2^2 + \ldots + m_s^2$$

in positive integers satisfying the conditions

$$\lambda_i n - A_i' n^{1-\beta} < m_i^2 \leqslant \lambda_i n + A_i n^{1-\beta} \qquad (i = 1, 2, \ldots, s).$$

This is an immediate consequence of Theorem 1 and Lemmas 3 and 8.

THEOREM 3. *If $s \geqslant 5$, $0 < \beta < \alpha$, and $n \geqslant 1$, there is a solution of the equation*

$$n = m_1^2 + m_2^2 + \ldots + m_s^2$$

in integers such that

$$|\lambda_i n - m_i^2| = O(n^{1-\beta}) \qquad (i = 1, 2, \ldots, s).$$

For n greater than some $n_0 = C$ this follows from Theorem 2. If

$$1 \leqslant n \leqslant C,$$

n can always be expressed as the sum of s squares, some of which may be zero. Also,

$$|\lambda_i n - m_i^2| \leqslant 2n < C = O(n^{1-\beta}).$$

THEOREMS ON POLYNOMIALS IN A GALOIS FIELD

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1. LET p be a prime and let $\xi(y)$ be a polynomial

$$a_0+a_1y+a_2y^2+\ldots+a_my^m,$$

where $a_0, a_1, a_2, \ldots, a_m$ are integers lying in the range 0, 1, 2,..., $p-1$, and a_m does not vanish except possibly in the case $m = 0$. We say that $\xi(y)$ is a polynomial reduced modulo p, and of degree* m, and we identify $\xi(y)$ with polynomials of the form

$$a_0+a_1y+a_2y^2+\ldots+a_my^m+0.y^{m+1}+0.y^{m+2}+\ldots+0.y^{m+k}.$$

Suppose that $\xi(y)$, $\eta(y)$ are the two polynomials $a_0+a_1y+\ldots+a_my^m$ and $b_0+b_1y+\ldots+b_my^m$ (we may assume, in virtue of what has already been said, that they have the same number of terms). We write

$$\xi(y)\pm\eta(y) = c_0+c_1y+\ldots+c_my^m,$$

where c_r lies in the range 0, 1, 2,..., $p-1$, and $c_r \equiv a_r \pm b_r \pmod p$, $(r = 0, 1, \ldots, m)$. We write $-\xi(y) = 0-\xi(y)$, and observe that when $p = 2$, then $\xi(y) \equiv -\xi(y)$.

We define $\xi(y)\eta(y)$ similarly in the following way. We write

$$\xi(y)\eta(y) = d_0+d_1y+\ldots+d_{2m}y^{2m},$$

where d_r lies in the range considered, and

$$d_r \equiv a_0b_r+a_1b_{r-1}+\ldots+a_rb_0 \pmod p.$$

If ξ and η are of degree m and n respectively, it is not difficult to see that the degree of $\xi\eta$ must be $m+n$.

The theory of polynomials of the type defined, and of the more general class of polynomials with coefficients in a finite Galois field, which we discuss later, has been systematically carried out, and theorems analogous to those of ordinary algebra may be obtained.†

We define as irreducible a polynomial $\pi(y)$ of degree $h > 0$, which cannot be expressed as a product of two polynomials each of degree greater than 0.

* The degree of the polynomial 0 is not defined.

† See, e.g., L. E. Dickson, *Linear Groups, with an Exposition of the Galois Field Theory*, 1901, pp. 1–54.

Now let $\pi(y)$ be an irreducible polynomial of degree h, and let $\Phi(y)$ be a rational integral polynomial. We can write

$$\Phi(y) = \pi(y)\lambda(y)+a_0+a_1y+a_2y^2+\ldots+a_{h-1}y^{h-1},$$

where $\lambda(y)$ is a polynomial of the field, and the remainder is a polynomial of degree $h-1$ with coefficients in the required range. This class of polynomials (of order p^h) form the residue class mod $\pi(y)$ for polynomials of the field considered. We call them the marks of the finite Galois field $GF(p^h)$. We denote by ϵ a mark of the field.

Now let $\theta(x)$ be a polynomial with coefficients ϵ lying in $GF(p^h)$. Write

$$\theta(x) = \epsilon_0+\epsilon_1x+\epsilon_2x^2+\ldots+\epsilon_mx^m.$$

If ϵ_m does not vanish, the polynomial $\theta(x)$ is said to be of degree* m.

We may now define addition, subtraction, and multiplication for our new polynomials $\theta(x)$. Let

$$\theta^{(1)}(x) = \epsilon_0^{(1)}+\epsilon_1^{(1)}x+\epsilon_2^{(1)}x^2+\ldots+\epsilon_m^{(1)}x^m,$$
$$\theta^{(2)}(x) = \epsilon_0^{(2)}+\epsilon_1^{(2)}(x)+\epsilon_2^{(2)}x^2+\ldots+\epsilon_m^{(2)}x^m;$$

we may again assume that $\theta^{(1)}$, $\theta^{(2)}$ are of the same degree. Then

$$\theta^{(1)}(x)+\theta^{(2)}(x) = \epsilon_0+\epsilon_1x+\epsilon_2x+\ldots+\epsilon_mx^m,$$

where $\epsilon_0, \epsilon_1, \ldots, \epsilon_m$ are marks of the field, and

$$\epsilon_r \equiv \epsilon_r^{(2)}+\epsilon_r^{(2)} \pmod{\pi(y), \text{ mod } p}.$$

And

$$\theta^{(1)}(x)\theta^{(2)}(x) = \epsilon_0'+\epsilon_1'x+\epsilon_2'x+\ldots+\epsilon_m'x^m,$$

where $\epsilon_0', \epsilon_1', \ldots, \epsilon_m'$ are marks of the field, and

$$\epsilon_r' \equiv \epsilon_0^{(1)}\epsilon_r^{(2)}+\epsilon_1^{(1)}\epsilon_{r-1}^{(2)}+\ldots+\epsilon_r^{(2)}\epsilon_0^{(2)} \pmod{\pi(y), \text{ mod } p}.$$

We observe that

$$\text{degree } \theta^{(1)}(x)\theta^{(2)}(x) = \text{degree } \theta^{(1)}(x) + \text{degree } \theta^{(2)}(x).$$

We define as irreducible a polynomial $\varpi(x)$ of degree greater than 0 which cannot be expressed as the product of two polynomials of degree greater than 0.

We use Greek letters generally to denote polynomials, and the letter ϖ to denote an irreducible polynomial. We call primary a polynomial when the coefficient of the highest power of x is the mark 1. It may be shown that every polynomial of degree greater than 0 may be expressed uniquely as the product of primary irreducible polynomials and one mark of the field (which, of course, is 1 when the original polynomial is primary).

* Again, when every coefficient ϵ vanishes, the degree is not defined.

Our first theorem concerns the number of ways in which a polynomial can be expressed as a sum of two nth powers. We prove that, given n and m, we can always find polynomials which can be expressed in m different ways as the sum of two nth powers.

We next go on to consider the analogue of Waring's theorem. The proof of Waring's theorem is not difficult, and furthermore in certain exceptional cases we are able to obtain some quite remarkable Waring numbers, provided that we admit the algebraic sum of powers of not necessarily primary polynomials taken positively or negatively.

2. THEOREM I. *Let n, m be two arbitrary positive integers. Then we can choose a polynomial ν which can be expressed in at least m different ways as the sum the nth powers of two primary polynomials.*

We observe that we may assume without loss of generality that n is divisible by p. For if ν can be expressed in m different ways in the form

$$\nu = \alpha^n + \beta^n, \tag{2.1}$$

then $\nu^p = \alpha^{np} + \beta^{np}$ for all the solutions of (2.1) and conversely, and hence ν^p can be expressed in m different ways as the sum of two (np)th powers. Next we observe that we may assume that n is of the form $p^k - 1$, for we can always find a polynomial of this form divisible by any n prime to p.

We use the identity

$$\{\xi(\xi^{p^k-1}+\eta^{p^k-1})\}^{p^k-1}+\{\eta(\xi^{p^k-1}+\eta^{p^k-1})\}^{p^k-1} = (\xi^{p^k-1}+\eta^{p^k-1})^{p^k} = (\xi^{p^k})^{p^k-1}+(\eta^{p^k})^{p^k-1}. \tag{2.2}$$

We observe that if the degrees of ξ and η are unequal, and if ξ, η are primary, then the two polynomials on the left-hand side of (2.2) are also primary. Now suppose that it has been established that we can find a polynomial ν, which can be expanded in N different ways as the sum of two (p^k-1)th powers

$$\nu = \alpha_1^{p^k-1}+\beta_1^{p^k-1} = \alpha_2^{p^k-1}+\beta_2^{p^k-1} = \ldots = \alpha_N^{p^k-1}+\beta_N^{p^k-1}, \tag{2.3}$$

such that the numbers $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_{N-1}, \beta_{N-1}$ are all primary p^kth powers, while $\alpha_N = \gamma\beta_N$, γ being an arbitrary primary polynomial which is not a p^kth power, and α_N, β_N also being primary. If in (2.2) we write $\xi = \gamma$, $\eta = 1$, then we observe that, for $N = 2$, the equation (2.3) is satisfied by putting

$$\alpha_1 = \gamma^{p^k}, \qquad \beta_1 = 1, \qquad \alpha_2 = \gamma(\gamma^{p^k-1}+1), \qquad \beta_2 = \gamma^{p^k-1}+1.$$

If now we put

$$\alpha_r' = \alpha_r^{p^k}, \qquad \beta_r' = \beta_r^{p^k} \qquad (r = 1, 2, \ldots, N),$$

then

$$\nu^{p^k} = \alpha_1'^{p^k-1}+\beta_1'^{p^k-1} = \alpha_2'^{p^k-1}+\beta_2'^{p^k-1} = \ldots = \alpha_N'^{p^k-1}+\beta_N'^{p^k-1},$$

while

$$\begin{aligned}\alpha_N'^{p^k-1}+\beta_N'^{p^k-1} &= (\alpha_N'^{p^k})^{p^k-1}+(\beta_N'^{p^k})^{p^k-1}\\ &= \{\alpha_N(\alpha_N'^{p^k-1}+\beta_N'^{p^k-1})\}^{p^k-1}+\{\beta_N(\alpha_N'^{p^k-1}+\beta_N'^{p^k-1})\}^{p^k-1}\\ &= \alpha_{N+1}'^{p^k-1}+\beta_{N+1}'^{p^k-1}.\end{aligned}$$

We observe that $\alpha_{N+1}' = \gamma\beta_{N+1}'$, so that the pair $\alpha_{N+1}', \beta_{N+1}'$ is certainly distinct from the pairs α_r', β_r' $(1 \leqslant r \leqslant N)$, the latter being all p^kth powers. Also all the numbers α_r', β_r' $(1 \leqslant r \leqslant N+1)$ are primary. Thus, if (2.3) is satisfied for $N = M$, then, by appropriate choice of ν, it may be satisfied for $N = M+1$. Finally, since it is satisfied with $N = 2$, the desired result follows.

If in (2.2) we make η primary and keep it fixed, and allow ξ to run through all primary polynomials of degree different to that of η, then we obtain the theorem

THEOREM II. *We can obtain an infinite number of primary polynomials which can be expressed in an infinite number of ways as the sum of the nth powers of three primary polynomials.*

3. Before discussing Waring's problem we need some preliminary lemmas.

LEMMA 1.

$$\begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & . & . & . & \alpha_1^{n-1}\\ 1 & \alpha_2 & \alpha_2^2 & . & . & . & \alpha_2^{n-1}\\ 1 & \alpha_3 & \alpha_3^2 & . & . & . & \alpha_3^{n-1}\\ & . & . & . & . & . & .\\ 1 & \alpha_n & \alpha_n^2 & . & . & . & \alpha_n^{n-1}\end{vmatrix} = \prod_{1\leqslant m<l\leqslant n}(\alpha_l-\alpha_m).$$

The lemma is immediate.

LEMMA 2. *Let $n = p^{n_1}+p^{n_2}+\ldots+p^{n_k}$, where none of the numbers $n_1, n_2, \ldots, n_k$ occurs more than $p-1$ times. Let $h(p-1)k' > k$. Then*

$$\sum_{\substack{\text{degree } \nu = k'\\ \nu \text{ primary}}} \nu^n = 0. \tag{3.1}$$

We observe first that in order to establish (3.1) it is sufficient to consider the same function summed over all polynomials of degree less than or equal to k' and with constant term unity. For we

obtain* this set of polynomials ν if we change x into $1/x$ and multiply by $x^{k'}$, so that the general term of the sum (3.1), which is

$$(x^{k'}+\epsilon_1 x^{k'-1}+\ldots+\epsilon_{k'})^n,$$

becomes

$$(\epsilon_{k'} x^{k'}+\epsilon_{k'-1} x^{k'-1}+\ldots+\epsilon_1 x+1)^n,$$

where all or any of the marks $\epsilon_1, \epsilon_2, \ldots, \epsilon_{k'}$ may vanish.

We prove the lemma by induction. It may easily be verified for $k'=1$. Now suppose that the lemma has already been established for $k'=m$, and consider a number n for which the least value of k' is $m+1$. Then

$$\sum_{\substack{\text{degree } \nu \leqslant m+1 \\ \text{constant term}=1}} \nu^n = \sum_{\substack{\text{degree } \nu \leqslant m \\ \text{constant term}=1}} \Big\{ \sum_{\epsilon} (\nu+\epsilon x^{m+1})^n \Big\},$$

where the internal summation is over all the marks of the field. The last sum may be written

$$\sum_{\substack{\text{degree } \nu \leqslant m \\ \text{constant term}=1}} \Big\{ \sum_{r=0}^{n} {}^n e_r \sum_{\epsilon} \epsilon^r x^{r(m+1)} \nu^{n-r} \Big\},$$

where ${}^n e_r$ denotes the residue (mod p) of ${}^n C_r$. Now the coefficient of $x^{r(m+1)}$ in the above sum is

$${}^n e_r \sum_{\substack{\text{degree } \nu \leqslant m \\ \text{constant term}=1}} \nu^{n-r} \sum_{\epsilon} \epsilon^r, \tag{3.2}$$

and vanishes unless $r>0$ and p^h-1 is a factor of r. Now

$$n = p^{n_1}+p^{n_2}+\ldots+p^{n_k}; \tag{3.3}$$

let us suppose that

$$r = p^{l_1}+p^{l_2}+\ldots+p^{l_{k_1}},$$
$$n-r = p^{m_1}+p^{m_2}+\ldots+p^{m_{k_2}},$$

and that neither of the sets $(l_1, l_2, \ldots, l_{k_1})$, $(m_1, m_2, \ldots, m_{k_2})$ contains the same number more than $p-1$ times, while $k_1+k_2>k$. Then some integer must be contained at least p times in the set $(l_1, l_2, \ldots, l_{k_1}, m_1, m_2, \ldots, m_{k_2})$. Let us suppose, as we may, that

$$l_1 = l_2 = \ldots = l_t = m_1 = m_2 = \ldots = m_{p-t},$$

and let us write

$$r' = p^{l_1+1}+p^{l_{t+1}}+p^{l_{t+2}}+\ldots+p^{l_{k_1}}$$
$$n-r' = p^{m_{p-t+1}}+p^{m_{p-t+2}}+\ldots+p^{m_{k_2}}.$$

Now the number of times that p is contained as a factor in the product $r!$ is

$$\sum_{s=1}^{k_1} (p^{l_s-1}+p^{l_s-2}+\ldots+1),$$

* The underlying idea, which is due to Mr. Davenport, will be developed elsewhere.

with similar expressions for the number of times that p occurs as a factor of $(n-r)!$, $r'!$, and $(n-r')!$. Thus the number of times that p occurs as a factor in $r'!(n-r')!$ exceeds the number of times that it occurs as a factor of $r!(n-r)!$ by

$$\begin{aligned} &p^{l_1}+p^{l_1-1}+\ldots+1+ \\ &+\sum_{s=l+1}^{k_1}(p^{l_s-1}+p^{l_s-2}+\ldots+1)+\sum_{s=p-l+1}^{k_2}(p^{m_s-1}+p^{m_s-2}+\ldots+1)- \\ &-\sum_{s=1}^{k_1}(p^{l_s-1}+p^{l_s-2}+\ldots+1)-\sum_{s=1}^{k_2}(p^{m_s-1}+p^{m_s-2}+\ldots+1) \\ &\qquad = p^{l_1}+p^{l_1-1}+\ldots+1-p(p^{l_1-1}+p^{l_1-2}+\ldots+1) \\ &\qquad = 1. \end{aligned}$$

The number of times that p occurs as a factor of $n!$ is not less than the number of times it occurs as a factor of $r'!(n-r')!$, and thus exceeds the number of times it occurs as a factor of $r!(n-r)!$. Thus we have shown that ${}^{n}e_{r}$ vanishes unless $k_1+k_2=k$.

Now since the third factor of (3.2) vanishes unless p^h-1 is a factor of r, and this implies that k_1 must be divisible by $h(p-1)$, it follows that (3.2) vanishes unless $k_2 \leqslant k-h(p-1)$. But this means that the corresponding value of k' can be at most m, and, since the lemma is supposed to have been established for $k'=m$, it follows that, when $k_2 \leqslant k-h(p-1)$, the second factor of (3.2) must vanish. Thus one of the three factors of (3.2) must always vanish, and this proves the lemma.

LEMMA 3.
$$\sum_{\substack{\text{degree } \nu=k' \\ \nu \text{ primary}}} \nu^{p^{hk'}-1} \not\equiv 0. \tag{3.4}$$

Suppose that the sum (3.4) vanished. Then we should have, by Lemma 2,
$$\sum_{\substack{\text{degree } \nu=k' \\ \nu \text{ primary}}} \nu^{m} = 0 \qquad (0 \leqslant m \leqslant p^{hk'}-1),$$
and thus the determinant of Lemma 1 would vanish with $n=p^{hk'}$, and α running through the primary polynomials of degree ν. This would give a contradiction.

4. Suppose that ϖ is an irreducible polynomial of degree n. Then it is known* that, for any α, $\alpha^{p^{hn}-1}$ must be congruent to 0 or 1 (mod ϖ). It follows that the sum of any number of $(p^{hn}-1)$th powers can only be congruent to 0, 1, 2,..., $p-1$ (mod ϖ). If, there-

* Dickson, loc. cit., Chap. I.

fore, we ask what is the smallest number of $(p^{hn}-1)$th powers as the sum of which all polynomials can be expressed, the question has clearly no finite answer. We can, however, prove the following theorems.

THEOREM III. *Let n be any integer prime to p. Then all polynomials which are not barred by congruences from being the sum of nth powers can be expressed as the sum of the nth powers of at most K polynomials, where $K = K(n)$ is a number which depends only on n.*

It will be sufficient to prove that all polynomials of the form $\Gamma\nu+\Delta$ (ν arbitrary) can be expressed as the sum of at most k nth powers, where Γ, Δ are polynomials which depend only on n, and k also depends only on n. For this will show first that the only congruences relevant to sums of nth powers can be those (mod Γ). Suppose that the possible residues (mod Γ) are $\alpha_1, \alpha_2, \ldots, \alpha_M$. Then it is possible to find polynomials ν_m $(1 \leqslant m \leqslant M)$ such that each of the polynomials $\Gamma\nu_m-\Delta+\alpha_m$ $(1 \leqslant m \leqslant M)$ can be expressed as the sum of nth powers. Let $k' = k'(n)$ be the maximum number of nth powers required for the expression of the finite set $\Gamma\nu_m-\Delta+\alpha_m$ $(1 \leqslant m \leqslant M)$. Since $\Gamma(\nu-\nu_m)+\Delta$ can be expressed as the sum of at most k nth powers, it follows that $\Gamma\nu+\alpha_m$ $(1 \leqslant m \leqslant M)$ can be expressed as the sum of at most $k+k'$. Since the last expression includes all polynomials which can be expressed as the sum of nth powers at all, the desired theorem follows.

To establish the assertion made above, we consider sums of the form

$$\sum_{\substack{\text{degree } \alpha = l \\ \alpha \text{ primary}}} (\alpha^{m_1}\nu+\alpha^{m_2})^n, \tag{4.1}$$

where l, m_1, and m_2 $(m_2 > m_1)$ are positive integers which remain to be chosen (m_1 may vanish). We first choose m_2, l so that

$$(n-1)m_2 < p^{hl} < nm_2,$$

and write

$$p^{hl}-1 = (n-1)m_2+m_1.$$

Now in the sum (4.1) the coefficient of ν^r $(r > 1)$ is of the form

$${}^ne_r \sum_{\substack{\text{degree } \alpha = l \\ \alpha \text{ primary}}} \alpha^{r'} \qquad (r' < p^{hl}-1),$$

and so vanishes by Lemma 2. On the other hand, the coefficient of ν is

$$\sum_{\substack{\text{degree } \alpha = l \\ \alpha \text{ primary}}} \alpha^{p^{hl}-1},$$

which by Lemma 3 is different from zero. Thus the sum (4.1) is of

the form $\Gamma\nu+\Delta$, and Theorem III then follows in virtue of what has already been said.

THEOREM IV. *Let n be prime to p. Then all polynomials which are not barred by congruences from being the sum of nth powers can be expressed as the sum of at most K' primary polynomials, where $K' = K'(n)$ is a number which depends only on n.*

Let Γ denote the number of Theorem III. Let γ be expressible as the sum of nth powers. In virtue of Theorem III we may suppose that

$$\gamma = \beta_1^n+\beta_2^n+\ldots+\beta_k^n.$$

Then we can choose a polynomial ξ such that $\xi\Gamma$ is primary and of degree sufficiently large to ensure that $\xi\Gamma+\beta_1$, $\xi\Gamma+\beta_2,\ldots,$ $\xi\Gamma+\beta_k$ are also all primary. Since

$$\gamma' = (\xi\Gamma+\beta_1)^n+(\xi\Gamma+\beta_2)^n+\ldots+(\xi\Gamma+\beta_k)^n \tag{4.2}$$

is congruent to $\gamma \pmod{\Gamma}$, we may write

$$\gamma-\gamma' = \Gamma(\nu_1+\nu_2+\ldots+\nu_p),$$

where the numbers $\nu_1, \nu_2,\ldots, \nu_p$ are all primary, and of degree sufficiently large to ensure that all the terms of the sum (4.1) with $\nu = \nu_1, \nu_2,\ldots, \nu_p$ are primary. Then

$$\gamma = \gamma'+\sum_{l=1}^{p}(\Gamma\nu_l+\Delta),$$

and, substituting for γ' and $(\Gamma\nu_l+\Delta)$ by means of (4.2) and (4.1) respectively, we obtain the desired result.

5. We will now investigate further the case in which $n = p^{hm}+1$. We observe first that

$$(\nu+x^{p^{hm}})^{p^{hm}+1}-(x\nu+1)^{p^{hm}+1}-\nu^{p^{hm}+1}+(x\nu)^{p^{hm}+1}$$
$$= \nu(x^{p^{2hm}}-x)+x^{p^{2hm}+p^{2hm}}-1 = \Gamma\nu+\Delta,$$

where, in virtue of a known theorem,*

$$\Gamma = \prod_{\substack{\text{degree } \varpi \mid 2m \\ \varpi \text{ irreducible primary}}} \varpi. \tag{5.1}$$

Thus, if we allow plus and minus signs arbitrarily, we have established the assertion made at the beginning of Theorem III, with Γ of the form (5.1) and $k = 4$. We will show that, with the notation of that theorem, if we still allow plus or minus signs, then $k' = 1$ or 2. We first need some preliminary lemmas.

* Dickson, loc. cit., pp. 15–16.

LEMMA 4. *Let ϖ be one of the factors of* (5.1) *and let degree $\varpi = r$. Then the highest common factor of $p^{hr}-1$ and $p^{hm}+1$ is 2 when $p > 2$, $r|m$, and of the form $(p^{hr}-1)/(p^{hr'}-1)$ otherwise.*

First suppose that r is a divisor of n. Then the highest common factor is clearly 2 when $p > 2$, and 1 when $p = 2$. If r is not a factor of n, then $r = 2r'$ where r' divides m and m/r' is odd. Then the highest common factor is $p^{hr'}+1 = (p^{hr}-1)/(p^{hr'}-1)$.

LEMMA 5. *Let $l < p^{hr}-1$. Then*

$$\sum_{\substack{\text{degree } \alpha \leq r-1 \\ \alpha \text{ not necessarily primary}}} \alpha^l = 0.$$

The proof is similar to that of Lemma 2.

LEMMA 6. *Let ϖ be an irreducible polynomial of degree r. Then we can find a polynomial α of lower degree than ϖ ($\alpha \neq 0$), such that*

$$\alpha^l \not\equiv 1 \pmod{\varpi} \qquad (l = 1, 2, \ldots, p^{hr}-2).$$

Suppose the lemma were not true. Then, as with ordinary algebra, we could show that there exists a number $l < p^{hr}-1$, such that

$$\alpha^l \equiv 1 \pmod{\varpi} \qquad (\text{all } \alpha, (\varpi, \alpha) = 1).$$

Thus addition gives

$$\sum_{\substack{\text{degree } \alpha \leq r-1 \\ \alpha \text{ not necessarily primary}}} \alpha^l \equiv -1 \pmod{\varpi},$$

which contradicts Lemma 5.

LEMMA 7. *Let ϖ be an irreducible polynomial of degree r. Let $ll' = p^{hr}-1$. Then the necessary and sufficient condition that the congruence*

$$\mu^l \equiv \alpha \pmod{\varpi}$$

is soluble, is

$$\alpha^{l'+1} \equiv \alpha \pmod{\varpi}.$$

Let α be the residue of the last lemma. Then the set $\{\alpha^l\}$ ($l = 1, 2, \ldots, p^{hr}-1$) (mod ϖ) runs through all the residues (except 0) (mod ϖ). Lemma 7 now follows by the familiar arguments of ordinary algebra.

LEMMA 8. *Let ϖ be one of the factors of* (5.1), *and let degree $\varpi = r$, where, unless $p = 2$, r is not a divisor of m. Then the necessary and sufficient condition that the congruence*

$$\mu^{p^{hm}+1} \equiv \beta \pmod{\varpi}$$

is soluble for some polynomial μ, is

$$\beta^{p^{hr'}} \equiv \beta \pmod{\varpi}, \tag{5.2}$$

where r' denotes the number of Lemma 4.

In the case where $p = 2$ or when r does not divide m, (5.3) follows at once from Lemma 7.

Since $$(\beta_1 \pm \beta_2)^{p^{hr'}} = \beta_1^{p^{hr'}} \pm \beta_2^{p^{hr'}},$$

(5.2) is also the condition that the polynomial β is congruent (mod ϖ) to a sum of nth powers.

Now let $p = 2$. We prove the following theorem.

THEOREM V. *Let $p = 2$. Then, with the notation of Theorem III, $K(2^{hm}+1) \leqslant 5$.*

Let γ be an arbitrary residue (mod Γ), which can be expressed (mod Γ) as the sum of nth powers. Let $\varpi_1, \varpi_2, \ldots, \varpi_N$ denote the factors of (5.1), and let $\gamma \equiv \beta_l \pmod{\varpi_l}$ $(l = 1, 2, \ldots, N)$. We can now find polynomials μ_l such that $\mu_l^n \equiv \beta_l \pmod{\varpi_l}$. Let $\mu \equiv \mu_l \pmod{\varpi_l}$ $(l = 1, 2, \ldots, N)$. Then $\mu^n \equiv \gamma \pmod{\Gamma}$, and thus $k' = 1$. This proves the theorem.

When $p > 2$, the result is not so elegant, since, when r is a divisor of m, the condition (5.2) now becomes equivalent to the condition that α should be a quadratic residue (mod ϖ).* We may, however, use the methods of Theorem V to prove the following theorem.

THEOREM VI. *Let $p > 2$. Let γ be a polynomial which can be expressed as the sum of nth powers, where n is prime to p, and $n = p^{hm}+1$. Let n' be the greatest odd factor of n. Then γ can be expressed as the algebraic sum of at most six nth powers taken positively or negatively, or as the sum of at most five n'th powers.*

Let $\varpi_1, \varpi_2, \ldots, \varpi_N$ denote the factors of (5.1), and let $\gamma \equiv \beta_l \pmod{\varpi_l}$ $(l = 1, 2, \ldots, N)$. If degree ϖ_l is not a divisor of m, we can find a polynomial μ_l such that $\mu_l^n \equiv \beta_l \pmod{\varpi_l}$. If degree ϖ_l is a divisor of m, then we can write $\beta_l \equiv \beta_l^{(1)} - \beta_l^{(2)} \pmod{\varpi_l}$, where $\beta_l^{(1)}$, $\beta_l^{(2)}$ are both quadratic residues (mod ϖ_l). We have only to observe that, since $p > 2$, we can write $2\eta_l^{(1)} \equiv \beta_l + 1 \pmod{\varpi_l}$ and $2\eta_l^{(2)} \equiv \beta_l - 1 \pmod{\varpi_l}$, and then $(\eta_l^{(1)})^2 - (\eta_l^{(2)})^2 \equiv \beta_l \pmod{\varpi_l}$, and we have only to put $\beta_l^{(1)} = (\eta_l^{(1)})^2$, $\beta_l^{(2)} = (\eta_l^{(2)})^2$. Now the equations $(\mu_l^{(1)})^n \equiv \beta_l^{(1)} \pmod{\varpi_l}$ and $(\mu_l^{(2)})^n \equiv \beta_l^{(2)} \pmod{\varpi_l}$ certainly have solutions. If finally we write

$$\begin{aligned}
&\mu^{(1)} \equiv \mu_l \pmod{\varpi_l} && \text{(degree } \varpi_l \text{ does not divide } m),\\
&\mu^{(1)} \equiv \mu_l^{(1)} \pmod{\varpi_l} && \text{(degree } \varpi_l \text{ divides } m),\\
&\mu^{(2)} \equiv 0 \pmod{\varpi_l} && \text{(degree } \varpi_l \text{ does not divide } m),\\
&\mu^{(2)} \equiv \mu_l^{(2)} \pmod{\varpi_l} && \text{(degree } \varpi_l \text{ divides } m),
\end{aligned}$$

* If $p = 2$, $h = 1$, then all polynomials are primary, and so $K'(2^m+1) \leqslant 5$.

then clearly $(\mu^{(1)})^n-(\mu^{(2)})^n \equiv \beta_l \pmod{\varpi_l}$ $(l = 1, 2, \ldots, N)$, and thus $(\mu^{(1)})^n-(\mu^{(2)})^n \equiv \gamma \pmod{\Gamma}$. It follows that $k' \leqslant 2$, and this proves the first part of Theorem VI.

For the second half of the theorem we observe that, when degree ϖ_l is not a divisor of m, then we can find a polynomial μ_l such that $\mu_l^{n'} \equiv \beta_l \pmod{\varpi_l}$, and that, when degree ϖ_l divides m, then the highest common factor of n' and $p^{hr}-1$ is 1, so that $\mu_l^{n'} \equiv \eta \pmod{\varpi_l}$ has a solution for all choices of η. Thus we can find a polynomial μ such that $\mu^{n'} \equiv \beta_l \pmod{\varpi_l}$ $(l = 1, 2, \ldots, N)$, or, what is the same thing, $\mu^{n'} \equiv \gamma \pmod{\Gamma}$. Thus we can express γ as the algebraic sum of 5 n'th powers in the form

$$\begin{aligned}\gamma &= \pm\xi_1^{n'} \pm \xi_2^{n'} \pm \xi_3^{n'} \pm \xi_4^{n'} \pm \xi_5^{n'} \\ &= (\pm\xi_1)^{n'} + (\pm\xi_2)^{n'} + (\pm\xi_3)^{n'} + (\pm\xi_4)^{n'} + (\pm\xi_5)^{n'},\end{aligned}$$

since n' is odd. Thus γ can be expressed as the sum of at most 5 n'th powers.

6. We confine ourselves now to the case where $p = 2$, $h = 1$. We prove the following theorem:

THEOREM VII. *Let $n > 1$. Then $K(n) \geqslant 3$.*

We may assume that n is odd. Let ϖ be an irreducible polynomial of degree r. Let $\alpha^n+\beta^n = \varpi$. We distinguish two cases.

(*a*) $\alpha+\beta = 1$.

In this case degree α = degree $\beta = l$ (say). Then it is not difficult to see that

$$\begin{aligned}\alpha^n+\beta^n &= \alpha^{n-1}+\alpha^{n-2}\beta+\ldots+\beta^{n-1} \\ &= \alpha^{n-1}+(\alpha+\beta)(\alpha^{n-3}+\alpha^{n-5}\beta+\ldots+\beta^{n-3})\beta. \qquad (6.1)\end{aligned}$$

The degree of the second member on the right-hand side does not exceed $(n-2)l$, while that of the first is $(n-1)l$. Thus

$$\text{degree } (\alpha^n+\beta^n) = (n-1)l,$$

and we get a contradiction if r is not divisible by $(n-1)$.

(*b*) $\alpha+\beta = \varpi$.

We must either have degree $\alpha = r$, degree $\beta < r$, in which case degree $(\alpha^n+\beta^n) = nr > r$, or else degree α = degree $\beta > r$ [since there are only two marks of the original Galois field, if α, β are both of degree r, it must follow that degree $(\alpha+\beta) < r$]. In the last case the argument used above shows that

$$\text{degree } (\alpha^n+\beta^n) = \text{degree } (\alpha+\beta)+(n-1)\text{ degree } \alpha > r.$$

Thus again we have a contradiction.

In the special case $n = 3$, we can sometimes improve on Theorem IV. The argument of that theorem gives

$$\Gamma = x^4+x = x(x+1)(x^2+x+1),$$

and the possible congruences of a cube with respect to these factors are

$$\alpha^3 \equiv 0 \text{ or } 1 \pmod{x}, \qquad \alpha^3 \equiv 0 \text{ or } 1 \pmod{x+1},$$

$$\alpha^3 \equiv 0 \text{ or } 1 \pmod{x^2+x+1}.$$

We prove the following theorem:

THEOREM VIII. *Let* $\gamma \equiv 0 \text{ or } 1 \pmod{x^2+x+1}, \gamma \not\equiv 0 \pmod{x^2+x}$. *Then* γ *can be expressed as the sum of four cubes.**

The possible congruences of $\gamma \pmod{x^4+x}$ are 1, x^2+x+1, x^3, x^3+1, x^3+x^2+x, x^3+x^2+x+1. We treat the six cases separately. We have only to observe that

$$(\nu x+x^2)^3+(\nu x+1)^3+(\nu(x+1)+x^2)^3+(\nu(x+1))^3 = (x^4+x)\nu+1$$

$$(\nu x+x^2+1)^3+(\nu x)^3+(\nu(x+1)+x^2)^3+(\nu(x+1))^3 = (x^4+x)(\nu+1)+x^2+x+1$$

$$(\nu(x+1)+x+1)^3+(\nu(x+1)+x)^3+(\nu+x^2+1)^3+\nu^3 = (x^4+x)(\nu+x^2+1)+x^3$$

$$(\nu(x+1)+1)^3+(\nu(x+1))^3+(\nu+x^2+1)^3+\nu^3 = (x^4+x)(\nu+x^2+1)+x^3+x^2+x$$

$$(\nu x+x)^3+(\nu x+x+1)^3+(\nu+x^2)^3+\nu^3 = (x^4+x)(\nu+x^2)+x^3+x^2+x+1$$

$$(\nu x+1)^3+(\nu x)^3+(\nu+x^2)^3+\nu^3 = (x^4+x)(\nu+x^2)+x^3+1.$$

* In this field all polynomials are primary.

ON THE FUNCTION $1/\zeta(1+it)$

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1. One of the unsolved problems in the theory of the Riemann zeta-function is that of determining how large are the greatest values which $|1/\zeta(1+it)|$ takes as $t \to \infty$. Writing

$$\mu = \overline{\lim_{t\to\infty}} \frac{|1/\zeta(1+it)|}{\log\log t}, \tag{1}$$

it was proved by Bohr and Landau* that μ is positive. It is not known that μ is finite; but, if the Riemann hypothesis is true, then μ is finite, and in fact

$$\mu \leqslant 2b, \tag{2}$$

where

$$b = \frac{6}{\pi^2} e^{\gamma}, \tag{3}$$

γ being Euler's constant. In view of this it is of some interest to try to determine a definite lower limit for μ.

The argument of Bohr and Landau of course leads to such a lower limit, but not to one of much interest. On the other hand, it was proved by Littlewood† that, on the Riemann hypothesis,

$$\mu \geqslant \tfrac{1}{8}b. \tag{4}$$

Here I prove without any hypothesis that

$$\mu \geqslant b. \tag{5}$$

Comparing (2) and (5), we see that (on the Riemann hypothesis) it is only a factor 2 which remains doubtful. The state of affairs in this problem is now exactly the same as in the corresponding problem for $\zeta(1+it)$, where similar results hold with $c = e^{\gamma}$ instead of b.

The method is suggested by my proof‡ of the corresponding theorem for $\zeta(1+it)$. That, however, depends on the fact that the coefficients in the Dirichlet series for $\zeta(s)$ are all positive. As this is not true for $1/\zeta(s)$, the details of the proof naturally follow a different course.

* Bohr and Landau (**1**) and (**2**). See the list of references at the end.
† Littlewood (**4**). ‡ Titchmarsh (**5**).

2. We first require some formulae.

Let k be a positive integer. Then, for $\sigma > 1$,

$$\frac{1}{\{\zeta(s)\}^k} = \prod_p \left(1-\frac{1}{p^s}\right)^k = \prod_p \left(1-\frac{k}{p^s}+\frac{k(k-1)}{1.2}\frac{1}{p^{2s}}+\ldots+\frac{(-1)^k}{p^{ks}}\right).$$

Hence
$$\frac{1}{\{\zeta(s)\}^k} = \sum_{n=1}^{\infty} \frac{b_k(n)}{n^s}, \tag{1}$$

where the coefficients $b_k(n)$ are determined in an obvious way from the preceding product.

The form of these coefficients shows that

$$\sum_{n=1}^{\infty} \frac{|b_k(n)|}{n^s} = \prod_p \left(1+\frac{k}{p^s}+\frac{k(k-1)}{1.2}\frac{1}{p^{2s}}+\ldots+\frac{1}{p^{ks}}\right)$$

$$= \prod_p \left(1+\frac{1}{p^s}\right)^k = \prod_p \left\{\left(1-\frac{1}{p^{2s}}\right)\Big/\left(1-\frac{1}{p^s}\right)\right\}^k = \left\{\frac{\zeta(s)}{\zeta(2s)}\right\}^k. \tag{2}$$

We have also to study the function

$$G_k(s) = \sum_{n=1}^{\infty} \frac{\{b_k(n)\}^2}{n^s}. \tag{3}$$

It is clear from the way in which the coefficients $b_k(n)$ are formed that, for $\sigma > 1$,

$$G_k(s) = \prod_p \left(1+\frac{k^2}{p^s}+\frac{k^2(k-1)^2}{1^2.2^2}\frac{1}{p^{2s}}+\ldots+\frac{1}{p^{ks}}\right)$$

$$= \prod_p g_k(p^{-s}), \tag{4}$$

say, where
$$g_k(z) = 1+k^2z+\frac{k^2(k-1)^2}{1^2.2^2}z^2+\ldots+z^k. \tag{5}$$

If z is real,
$$g_k(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\sum_{m=0}^{k} \frac{k!}{m!(k-m)!} z^{\frac{1}{2}m} e^{im\phi}\right|^2 d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} |1+z^{\frac{1}{2}}e^{i\phi}|^{2k}\, d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} (1+2z^{\frac{1}{2}}\cos\phi+z)^k\, d\phi \tag{6}$$

$$= \frac{(1-z)^k}{\pi}\int_0^{\pi} \left(\frac{1+z}{1-z}+\frac{2z^{\frac{1}{2}}}{1-z}\cos\phi\right)^k d\phi. \tag{7}$$

It follows from (7) and Laplace's integral for the Legendre polynomials that*

$$g_k(z) = (1-z)^k P_k\left(\frac{1+z}{1-z}\right), \tag{8}$$

where $P_k(z)$ is the Legendre polynomial of degree k. This formula of course holds for complex z also, and hence

$$G_k(s) = \prod_p \left\{(1-p^{-s})^k P_k\left(\frac{1+p^{-s}}{1-p^{-s}}\right)\right\}$$

$$= \frac{1}{\{\zeta(s)\}^k} \prod_p \left\{P_k\left(\frac{1+p^{-s}}{1-p^{-s}}\right)\right\}. \tag{9}$$

This formula is of some interest in itself, and leads to the identity

$$G_k(s) = F_{k+1}(s)/\{\zeta(s)\}^{2k+1}, \tag{10}$$

where†

$$F_k(s) = \sum_{n=1}^{\infty} \frac{\{d_k(n)\}^2}{n^s}.$$

Here, however, we can get what we want more conveniently by using (6) directly. If z is real and positive,

$$g_k(z) \leqslant \frac{1}{\pi} \int_0^{\pi} (1+z^{\frac{1}{2}})^{2k}\, d\phi = (1+z^{\frac{1}{2}})^{2k}. \tag{11}$$

Hence, for $\sigma > 2$,

$$G_k(\sigma) \leqslant \prod_p \left(1+\frac{1}{p^{\frac{1}{2}\sigma}}\right)^{2k} = \prod_p \left(\frac{1-p^{-\sigma}}{1-p^{-\frac{1}{2}\sigma}}\right)^{2k} = \left\{\frac{\zeta(\frac{1}{2}\sigma)}{\zeta(\sigma)}\right\}^{2k}. \tag{12}$$

On the other hand,

$$g_k(z) \geqslant \frac{1}{\pi} \int_0^{\pi/k} |1+z^{\frac{1}{2}}e^{i\phi}|^{2k}\, d\phi$$

$$= \frac{(1+z^{\frac{1}{2}})^{2k}}{\pi} \int_0^{\pi/k} \left\{1 - \frac{2z^{\frac{1}{2}}(1-\cos\phi)}{1+2z^{\frac{1}{2}}+z}\right\}^k d\phi.$$

Here

$$1-\cos\phi \leqslant 2\sin^2(\pi/2k) = O(k^{-2}), \qquad z^{\frac{1}{2}}/(1+2z^{\frac{1}{2}}+z) = O(1).$$

* The formula (8) is, essentially, Murphy's well-known formula

$$P_k(\cos\theta) = \cos^{2k}\tfrac{1}{2}\theta\, F(-k, -k; 1; -\tan^2\tfrac{1}{2}\theta)$$

with $z = -\tan^2\frac{1}{2}\theta$; cf. Hobson, *Spherical and Ellipsoidal Harmonics* (1931), pp. 22, 31.

† See Titchmarsh (5), pp. 74–5.

Hence
$$g_k(z) = \frac{(1+z^{\frac{1}{2}})^{2k}}{\pi}\int_0^{\pi/k}\left\{1+O\left(\frac{1}{k^2}\right)\right\}^k d\phi$$
$$> \frac{(1+z^{\frac{1}{2}})^{2k}}{2k} \tag{13}$$

for $k > k_0$, independently of z. Hence, for $\sigma > 1$,

$$G_k(\sigma) > \prod_{p\leqslant x}\left\{\frac{1}{2k}\left(1+\frac{1}{p^{\frac{1}{2}\sigma}}\right)^{2k}\right\}, \tag{14}$$

where x is any positive number.

3. We can now prove the main theorem. Let $\sigma > 1$, and let N be any positive number. Then

$$\frac{1}{T}\int_0^T\left|\sum_{n\leqslant N}\frac{b_k(n)}{n^s}\right|^2 dt = \frac{1}{T}\int_0^T\sum_{m\leqslant N}\frac{b_k(m)}{m^{\sigma+it}}\sum_{n\leqslant N}\frac{b_k(n)}{n^{\sigma-it}}\,dt$$
$$= \sum_{n\leqslant N}\frac{\{b_k(n)\}^2}{n^{2\sigma}} + \frac{1}{T}\sum\sum_{m\neq n}\frac{b_k(m)b_k(n)}{m^\sigma n^\sigma}\int_0^T\left(\frac{n}{m}\right)^{it}dt$$
$$\geqslant \sum_{n\leqslant N}\frac{\{b_k(n)\}^2}{n^{2\sigma}} - \frac{1}{T}\sum\sum_{m\neq n}\frac{|b_k(m)b_k(n)|}{m^\sigma n^\sigma}\,\frac{2}{|\log(n/m)|}.$$

Here
$$\left|\log\frac{n}{m}\right| \geqslant \left|\log\frac{n}{n+1}\right| \geqslant \frac{1}{2n} \geqslant \frac{1}{2N},$$

so that the last term is less than

$$\frac{4N}{T}\sum\sum_{m\neq n}\frac{|b_k(m)b_k(n)|}{m^\sigma n^\sigma} < \frac{4N}{T}\left(\sum_{n=1}^\infty\frac{|b_k(n)|}{n^\sigma}\right)^2 = \frac{4N}{T}\left\{\frac{\zeta(\sigma)}{\zeta(2\sigma)}\right\}^{2k}.$$

Since $\zeta(\sigma) \sim 1/(\sigma-1)$ as $\sigma \to 1$, we have, if σ is sufficiently near to 1,

$$\frac{\zeta(\sigma)}{\zeta(2\sigma)} < \frac{\frac{3}{2}}{\zeta(2)(\sigma-1)} = \frac{9}{\pi^2(\sigma-1)} < \frac{1}{\sigma-1},$$

and the above last term is less than

$$4N/T(\sigma-1)^{2k}.$$

Also
$$\left|\frac{1}{\{\zeta(s)\}^k} - \sum_{n\leqslant N}\frac{b_k(n)}{n^s}\right| \leqslant \sum_{n>N}\frac{|b_k(n)|}{n^s} < \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\sum_{n>N}\frac{|b_k(n)|}{n^{\frac{1}{2}\sigma+\frac{1}{2}}}$$
$$< \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\left\{\frac{\zeta(\frac{1}{2}\sigma+\frac{1}{2})}{\zeta(\sigma+1)}\right\}^k < \frac{1}{N^{\frac{1}{2}\sigma-\frac{1}{2}}}\left(\frac{2}{\sigma-1}\right)^k,$$

for σ sufficiently near to 1; and similarly

$$G_k(2\sigma)-\sum_{n\leqslant N}\frac{\{b_k(n)\}^2}{n^{2\sigma}}=\sum_{n>N}\frac{\{b_k(n)\}^2}{n^{2\sigma}}<\frac{1}{N^{\sigma-1}}\sum_{n>N}\frac{\{b_k(n)\}^2}{n^{\sigma+1}}$$

$$<\frac{1}{N^{\sigma-1}}G_k(\sigma+1)<\frac{1}{N^{\sigma-1}}\left\{\frac{\zeta(\frac{1}{2}\sigma+\frac{1}{2})}{\zeta(\sigma+1)}\right\}^{2k}<\frac{1}{N^{\sigma-1}}\left(\frac{2}{\sigma-1}\right)^{2k}.$$

These two differences are therefore both bounded if

$$N=\left(\frac{2}{\sigma-1}\right)^{2k/(\sigma-1)}.$$

With this value of N we have

$$\frac{1}{T}\int_0^T\left|\frac{1}{\{\zeta(s)\}^k}+O(1)\right|^2dt>G_k(2\sigma)-\frac{4N}{T}\left(\frac{1}{\sigma-1}\right)^{2k}+O(1)$$

$$>\prod_{p\leqslant x}\left\{\frac{1}{2k}\left(1+\frac{1}{p^\sigma}\right)^{2k}\right\}-\frac{4N}{T}\left(\frac{1}{\sigma-1}\right)^{2k}+O(1),$$

by (14) of § 2. The number of terms in this product is $\pi(x)<Ax/\log x$. Hence, if $x>\sqrt{k}$,

$$\prod_{p\leqslant x}\frac{1}{2k}\geqslant\left(\frac{1}{2k}\right)^{Ax/\log x}=\exp\left(-\frac{Ax\log 2k}{\log x}\right)>e^{-Ax}.$$

Also

$$\log\prod_{p\leqslant x}\left(\frac{1+p^{-1}}{1+p^{-\sigma}}\right)=\sum_{p\leqslant x}O\left(\frac{1}{p}-\frac{1}{p^\sigma}\right)=\sum_{p\leqslant x}O\left(\log p\int_1^\sigma\frac{du}{p^u}\right)$$

$$=O\left\{(\sigma-1)\sum_{p\leqslant x}\frac{\log p}{p}\right\}=O\{(\sigma-1)\log x\},$$

and* as $x\to\infty$

$$\prod_{p\leqslant x}\left(1+\frac{1}{p}\right)=\{b+o(1)\}\log x.$$

Hence

$$\prod_{p\leqslant x}\left\{\frac{1}{2k}\left(1+\frac{1}{p^\sigma}\right)^{2k}\right\}>e^{-Ax-Ak(\sigma-1)\log x}\{b+o(1)\}^{2k}\log^{2k}x.$$

Let $x=\delta k$, where $k^{-\frac{1}{2}}<\delta<1$, and $\sigma=1+\eta/\log k$, where $0<\eta<1$. Then

$$N\left(\frac{1}{\sigma-1}\right)^{2k}<\left(\frac{2\log k}{\eta}\right)^{(2k\log k)/\eta+2k},$$

* Littlewood (4), p. 350.

and we obtain

$$\frac{1}{T}\int_0^T \left|\frac{1}{\{\zeta(s)\}^k}+O(1)\right|^2 dt$$

$$> e^{-A\delta k - A\eta k}\{b+o(1)\}^{2k}\log^{2k}\delta k - \frac{4}{T}\left(\frac{2\log k}{\eta}\right)^{(2k\log k)/\eta+2k}+O(1).$$

Finally, let $$T=\left(\frac{2\log k}{\eta}\right)^{(2k\log k)/\eta+2k}.$$

Then $$\log T=\left(\frac{2k\log k}{\eta}+2k\right)\log\frac{2\log k}{\eta},$$

$$\operatorname{loglog} T=\log k+\log\left(\frac{2\log k}{\eta}+2\right)+\operatorname{loglog}\frac{2\log k}{\eta}$$

$$<(1+\epsilon)\log k$$

for $k>k_1=k_1(\epsilon,\eta)$. Hence

$$\frac{1}{T}\int_0^T \left|\frac{1}{\{\zeta(s)\}^k}+O(1)\right|^2 dt$$

$$> e^{-A\delta k - A\eta k}\{b+o(1)\}^{2k}\left(\frac{\operatorname{loglog} T}{1+\epsilon}-\log\frac{1}{\delta}\right)^{2k}+O(1).$$

4. Let $$M_{\sigma,T}=\max_{0\le t\le T}|1/\zeta(\sigma+it)|.$$

Since the first term on the right of the above inequality tends to infinity with k (for fixed δ, η, and ϵ), it is clear that $M_{\sigma,T}^k$ tends to infinity; hence

$$\left|\frac{1}{\{\zeta(s)\}^k}+O(1)\right|<2M_{\sigma,T}^k,$$

if k is large enough, and we deduce that

$$4M_{\sigma,T}^{2k}>\tfrac{1}{2}e^{-A\delta k - A\eta k}\{b+o(1)\}^{2k}\left(\frac{\operatorname{loglog} T}{1+\epsilon}-\log\frac{1}{\delta}\right)^{2k},$$

for k large enough. Hence

$$M_{\sigma,T}>\left(\frac{1}{8}\right)^{1/2k}e^{-A\delta - A\eta}\{b+o(1)\}\left(\frac{\operatorname{loglog} T}{1+\epsilon}-\log\frac{1}{\delta}\right).$$

Giving δ, ϵ, and η arbitrarily small values, and then varying T, it is clear that

$$\overline{\lim}\,\frac{M_{\sigma,T}}{\operatorname{loglog} T}\geqslant b,$$

where, of course, σ is a function of T. But if $\mu < b$, it follows from the usual Phragmén-Lindelöf argument* that

$$|1/\zeta(\sigma+it)| < (\mu+\epsilon)\log\log t < (b-\epsilon)\log\log t$$

throughout the strip $1 \leqslant \sigma \leqslant 2$. This contradicts the result obtained, and the theorem therefore follows.

* See Littlewood (3), p. 193.

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SPATIAL DISTANCE IN GENERAL RELATIVITY

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A PROBLEM introduced recently is that of finding a suitable definition of spatial distance, i.e. the distance of a star from an observer, in a general Riemannian space-time. Professor E. T. Whittaker* gave a definition based on the method of comparing absolute with apparent brightness of the star, and this has been modified† to allow for the motion of the star. In this paper is introduced a similar definition, made possible by a new theorem on null geodesics. It can easily be verified that this definition is equivalent to that given by Etherington.

Relative coordinates are frequently used in this paper, these having already been introduced by the author.‡

1. Relative coordinates

If C is a curve in a Riemannian space V_n, and if an orthogonal ennuple is given at points of C, then at each point P of C is defined a system of normal coordinates with P as origin, the parametric directions at P in these coordinates being the directions of the vectors of the given ennuple at this point. Thus there is a system of reference at each point of the curve. These coordinates are called *relative coordinates*, and are written $(z^0, z^1, \ldots, z^{n-1})$. If the points of C are defined in terms of a parameter s, taken to be the arc if C is not null, any other curve C' may be given by the z's as functions of s, a $(1, 1)$ correspondence being set up between the points of C and C'. In general, we only consider curves in the neighbourhood of C, so that the third, and sometimes the second, powers of z may be neglected. It is usually convenient to take a particular correspondence, the most useful being such that $z^0 = 0$, i.e. if $\lambda_{\sigma|}{}^i$ $(\sigma, i = 0, 1, \ldots, n-1)$ are the components of the vectors of the given ennuple, the point Q of C' corresponding to P of C is the point where C' meets the geodesic surface orthogonal to the direction $\lambda_{0|}{}^i$ at P.

If x'^i are the given coordinates of V_n, the curve C is defined by equations of the form $x'^i = x^i(s)$, and if g_{ij} is the fundamental tensor,

* *Proc. Roy. Soc.*, A, 133 (1931), 93.

† Etherington (not yet published).

‡ *Proc. Roy. Soc. Edin.*, 52 (1932), 345.

the tangent vector of C is λ^i, where

$$\lambda^i = dx^i/ds, \qquad g_{ij}\lambda^i\lambda^j = e, \tag{1.1}$$

and $e = 0, \pm 1$ according as C is, or is not, null. Invariants u^σ, $\gamma_{\sigma\nu}$ are defined by the equations

$$\lambda^i = u^\sigma \lambda_{\sigma|}{}^i, \qquad \gamma_{\sigma\nu} = -\gamma_{\nu\sigma} = \lambda_{\sigma|i,j}\lambda_{\nu|}{}^i\lambda^j, \tag{1.2}$$

an index appearing as a subscript and also as a superscript indicating summation, unless attached to $e = \pm 1$, the summation being from 0 to $n-1$ unless otherwise stated. With this notation, the arc $\bar{s}$ of a curve C' near C is given by

$$\pm\left(\frac{d\bar{s}}{ds}\right)^2 = e + 2\gamma_{\sigma h}u^h z^\sigma + 2\sum_\sigma e_\sigma u^\sigma \dot{z}^\sigma + \sum_\sigma e_\sigma(\dot{z}^\sigma)^2 + 2\gamma_{\sigma\nu}z^\sigma\dot{z}^\nu + \\ + \sum_\sigma e_\sigma \gamma_{\sigma\mu}\gamma_{\sigma\nu}z^\mu z^\nu + \Gamma_{\sigma\nu}z^\sigma z^\nu, \tag{1.3}$$

where $\dot{z}^\sigma = dz^\sigma/ds, \qquad e_\sigma = g_{ij}\lambda_{\sigma|}{}^i\lambda_{\sigma|}{}^j = \pm 1,$

and $$\Gamma_{\sigma\nu} = \Gamma_{\nu\sigma} = R_{hijk}\lambda^h\lambda^k\lambda_{\sigma|}{}^i\lambda_{\nu|}{}^j. \tag{1.4}$$

If C is a geodesic, then, from (1.2),

$$\gamma_{\sigma h}u^h = \lambda_{\sigma|i,j}\lambda^i\lambda^j = e_\sigma \dot{u}^\sigma,$$

and hence $$\gamma_{\sigma h}u^h z^\sigma + \sum_\sigma e_\sigma u^\sigma \dot{z}^\sigma = \dot{J},$$

where $$J = \sum_\sigma e_\sigma u^\sigma z^\sigma. \tag{1.5}$$

If C is a null geodesic,* and C' a null curve, $e = 0$, and $d\bar{s}/ds = 0$, and from (1.3) we have

$$J = \text{const.} \tag{1.6}$$

to the first order of approximation.

2. A theorem on null geodesics

If P_0, P are points on a null geodesic C, let p denote the thin† pencil of ∞^{n-2} null geodesics through P_0 passing near P. We shall prove that *the volume of the $(n-2)$-dimensional cross-section of p by a surface orthogonal to a vector V^i at P is independent of the direction of V^i.*

* It is assumed that when C is a null geodesic, the parameter s is so chosen that the geodesic equations of C take the usual simplified form. The parameter is then defined except for arbitrary additive and multiplicative constants.

† There should be no ambiguity in the use of the word 'thin'. Although the inner product of the tangent vectors at P_0 of two members of the pencil is zero to the first order, the angle between their projections on a space not orthogonal to C at P_0 is small, i.e. the solid angle of the pencil as measured by an observer at P_0 is small.

The pencil p being thin, the cross-section is evidently independent of the particular surface orthogonal to V^i. It is assumed that the direction of V^i is such that the square of distances in the cross-section may be neglected. Referring to relative coordinates along C, we shall take $z^0 = 0$, as explained in § 1, and we shall show that the volume when $V^i = \lambda_{0|}{}^i$ is equal to the volume for any other cross-section.

The null geodesics belonging to the pencil p are given by equations of the form

$$z^r = z^r(s, \alpha^1, \alpha^2, \ldots, \alpha^{n-2}) \qquad (r = 1, 2, \ldots, n-1),$$

where $z^r = 0$ at P_0, and the α's are constants, varying for the different geodesics. A surface orthogonal to $\lambda_{0|}{}^i$ is $z^0 = 0$ in relative coordinates, and, writing $Dz^r = (\partial z^r/\partial \alpha^p) d\alpha^p$, the linear element of the cross-section at P by this surface is given by

$$\pm d\sigma^2 = \sum_{r=1}^{n-1} e_r (Dz^r)^2.$$

If the surface $z^0 = 0$ at P meets a null geodesic C' of p at the point Q, it has been shown that the tangent vector of C' at Q is μ^σ, where

$$\mu^\sigma = \frac{\delta z^\sigma}{ds} = u^\sigma + \frac{dz^\sigma}{ds} - e_\sigma \gamma_{\sigma\nu} z^\nu.$$

Hence, a point Q' of C', near Q, has coordinates $\bar{z}^\sigma = z^\sigma + \phi\mu^\sigma$, where ϕ is an arbitrary function of s, of the order of smallness of the z's. Choosing ϕ so that Q' is the point of intersection of C' with a surface orthogonal to a vector V^i at P, and neglecting second-order terms, the points of the cross-section of p by this surface are $\bar{z}^\sigma = z^\sigma + \phi u^\sigma$, where ϕ is now a function of s and the α's. The element of length of this cross-section is given by

$$\pm d\bar{\sigma}^2 = \sum_{\sigma=0}^{n-1} e_\sigma (D\bar{z}^\sigma)^2 = \sum_\sigma e_\sigma (Dz^\sigma + u^\sigma D\phi)^2.$$

We have $\sum e_\sigma (u^\sigma)^2 = 0$, and $\sum e_\sigma u^\sigma Dz^\sigma = 0$, the latter following from (1.6), in which the constant is zero as the z's all vanish at P_0. Hence

$$d\bar{\sigma}^2 = \pm \sum e_\sigma (Dz^\sigma)^2 = d\sigma^2.$$

Thus the linear element of the cross-section is independent of the particular cross-section through P, from which the theorem follows immediately.

It has been pointed out to me that this theorem is a direct generalization of a simple property of surfaces on a null cone in a

flat space. In space of the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

the null cone through the origin is given by

$$x = rl, \qquad y = rm, \qquad z = rn, \qquad t = r; \qquad l^2+m^2+n^2 = 1,$$

and a surface on this cone is given by an expression for r as a function of l, m, n. Its line element is

$$d\sigma^2 = -dr^2 + \sum (r\,dl + l\,dr)^2 = r^2 \sum dl^2,$$

since $\sum l^2 = 1$, $\sum l\,dl = 0$. Thus the elements of distance and area at any point are the same for all surfaces through this point.

3. The equations of geodesics

If C is a geodesic, not null, and the vectors $\lambda_{\sigma|}{}^i$ are given by parallel displacement along C, then $\gamma_{\sigma\nu} = 0$, and, from (1.3),

$$T = \frac{d\bar{s}}{ds} = 1 + e\dot{J} - \tfrac{1}{2}\dot{J}^2 + \tfrac{1}{2}e \sum_\sigma e_\sigma (\dot{z}^\sigma)^2 + \tfrac{1}{2}e\Gamma_{\sigma\nu} z^\sigma z^\nu. \tag{3.1}$$

Varying the z's as functions of s, the geodesics near C are given by $\delta \int T\,ds = 0$, i.e. by

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \dot{z}^\sigma}\right) - \frac{\partial T}{\partial z^\sigma} = 0 \qquad (\sigma = 0, 1, \ldots, n-1). \tag{3.2}$$

Substituting from (3.1), these equations become

$$\ddot{z}^\sigma - e_\sigma \Gamma_{\sigma\nu} z^\nu = e u^\sigma \ddot{J}.$$

If, as before, we take $z^0 = 0$, then

$$e u^0 \ddot{J} = -e_0 \Gamma_{0\nu} z^\nu,$$

and hence, if $\lambda_{0|}{}^i$ is not orthogonal to C, the geodesics are given by

$$\ddot{z}^r - \left(e_r \Gamma_{rs} - \frac{e_0}{u^0} u^r \Gamma_{0s}\right) z^s = 0 \qquad (r, s = 1, 2, \ldots, n-1) \tag{3.3}$$

to the first order in the z's.

We observe that these equations are unaltered when $e = 0$, although the method of obtaining them is not valid in this case. We should therefore expect (3.3) to be the equations of geodesics when C is null. This is true, and can be verified by an actual transformation of coordinates in V_n. In what follows, we take C to be a null geodesic, the vectors $\lambda_{\sigma|}{}^i$ being given by parallel transport along C. From a property of null vectors we may take the vectors $\lambda_{2|}{}^i, \ldots, \lambda_{n-1|}{}^i$ to be independent of λ^i, and orthogonal to C. In this case $u^2, \ldots, u^{n-1}$ vanish, and (1.6) becomes

$$z^1 = \text{const.} \tag{3.4}$$

From the identities $\Gamma_{\sigma\nu} u^\sigma = 0$, $\sum e_\sigma (u^\sigma)^2 = 0$, equation (3.3) for $r = 1$ is now satisfied identically, and the remaining equations give

$$\ddot{z}^p - e_p \Gamma_{pr} z^r = 0 \qquad (p = 2, 3, \ldots, n-1), \tag{3.5}$$

where r takes the values 1 to $n-1$. Hence, the null geodesics near C are given by equations (3.4), (3.5), together with $z^0 = 0$.

4. Spatial distance in general relativity

Let A be a star, B an observer, and, in order that light may pass from A to B, let A and B lie on a null geodesic. We shall adopt the following definition:

The spatial distance of A from B is proportional to the square root of the two-dimensional cross-section of a thin pencil of null geodesics issuing from A and passing near B, made by the three-dimensional instantaneous space of the observer. The definition is made complete by the requirement that when the observer is near and has the motion of the star, the spatial distance must reduce to the element of length in the observer's instantaneous space.

The instantaneous space of an observer is orthogonal to his direction of motion, and hence, from the theorem of § 2, the above cross-section is independent of the direction of motion of the observer. Thus we see that *spatial distance is independent of the observer's motion*. It follows, from the definition, that the formula for spatial distance will consist of two factors, the first being an invariant Θ depending only on the positions of the star and observer, the second being independent of the position of the observer. The main problem is to find the above invariant.

If λ^i is the tangent vector of the null geodesic C, let $\lambda_{2|}{}^i$, $\lambda_{3|}{}^i$ be two orthogonal vectors independent of λ^i, both being orthogonal to C and given by parallel transport along C. Then, from (3.4) and (3.5), the null geodesics through A passing near B are given by $z^1 = 0$, and

$$\ddot{z}^p + \Gamma_{pq} z^q = 0 \qquad (p, q = 2, 3), \tag{4.1}$$

where Γ_{pq} are defined by (1.4), and are functions of s.* We have assumed that the vectors are chosen so that $e_2 = e_3 = -1$, these vectors being supposed to lie in the instantaneous space of some observer at B. The problem therefore reduces to the solution of the simultaneous linear differential equations (4.1).

* It is assumed that the parameter s is chosen so that it increases when passing from A to B along the null geodesic.

If the position of A is given by $s = s_0$, it is required that the solutions of (4.1) should vanish when $s = s_0$. They can therefore be written in the form

$$z^2 = \alpha\phi_1 + \beta\phi_2, \qquad z^3 = \alpha\psi_1 + \beta\psi_2, \tag{4.2}$$

where ϕ_1, ϕ_2, ψ_1, ψ_2 are functions of s and s_0 vanishing at s_0, and α, β are small arbitrary constants. The cross-section of the pencil of null geodesics by the plane $z^0 = 0$ at B is given by equations (4.2) and $z^1 = 0$, the variables being α, β, and s having its value at B. The area of cross-section is at once found to be proportional to $(\phi_1\psi_2 - \phi_2\psi_1)$, and the required invariant Θ must be proportional to the square root of this function. We therefore define

$$\pm\Theta^2 = K(\phi_1\psi_2 - \phi_2\psi_1), \tag{4.3}$$

where the arbitrary constant K is chosen so that $\Theta \sim s - s_0$ as $s \to s_0$. Thus

$$\frac{1}{K} = \lim_{s \to s_0} \frac{\phi_1\psi_2 - \phi_2\psi_1}{(s - s_0)^2}.$$

From the definition, the spatial distance between the points s_0 and s is given by an equation of the form

$$\Delta = \nu\Theta, \tag{4.4}$$

where ν is independent of s. For an observer B' at the point $s_0 + \epsilon$, where ϵ is small, we have $\Theta = \epsilon$, from (4.3), and $\Delta' = \nu\epsilon$. If the star and the observer B' both have the motion v^i, the distance δ between the star and B' in the instantaneous space of this observer is

$$\delta = \epsilon |g_{ij} v^i \lambda^j|_{s=s_0}.$$

Hence, to satisfy the second part of the definition, we must have

$$\nu = |g_{ij} v^i \lambda^j|_{s=s_0}. \tag{4.5}$$

Thus the spatial distance between the star and the observer is $\Delta = \nu\Theta$, where ν is the inner product of the direction of motion of the star and the tangent vector of the null geodesic, and Θ is the invariant defined as above. We observe that Θ and ν are not completely determined by the positions of the star and observer and the motion of the star, owing to the arbitrary multiplicative constant attached to the parameter s. This constant, however, is cancelled in the product $\Delta = \nu\Theta$.

5. Spatial distance in space of constant curvature

In a space of constant curvature K_0, the components of the curvature tensor satisfy

$$R_{hijk} = K_0(g_{hj}g_{ik} - g_{hk}g_{ij}).$$

We have $g_{ij}\lambda^i\lambda^j = 0$, and $g_{ij}\lambda^i\lambda_{p|}{}^j = 0$ for $p = 2, 3$, and hence, from (1.4),

$$\Gamma_{pq} = 0 \qquad (p, q = 2, 3).$$

From (4.1), the null geodesics through s_0 are given by

$$z^2 = \alpha(s - s_0), \qquad z^3 = \beta(s - s_0),$$

and from (4.3) we have

$$\Theta = s - s_0.$$

Hence, *in a space of constant curvature, if the parameter s of the null geodesic is chosen to vanish at the position of the star, the spatial distance is proportional to s.**

6. A more general method of obtaining spatial distance

The method of § 4 depends on finding two solutions of the equations of parallel transport along the given null geodesic. In some cases this may be impracticable; it can be avoided as follows:

Let $\lambda_{2|}{}^i$, $\lambda_{3|}{}^i$ be two orthogonal vectors at points of the geodesic, both being orthogonal to the curve and satisfying $e_2 = e_3 = -1$. As before, we have $u^2 = u^3 = 0$, and $z^1 = 0$ for the null geodesics through A. Completing an orthogonal 4-uple at points of C, we find, by a method similar to that of § 3, that the null geodesics through A are given by

$$\ddot{z}^p + 2\gamma_{pq}\dot{z}^q + \rho_{pq}z^q = 0, \tag{6.1}$$

where

$$\begin{aligned}\rho_{pq} &= \Gamma_{pq} + \dot{\gamma}_{pq} + \sum_{\sigma=0}^{3} e_\sigma \gamma_{\sigma p}\gamma_{\sigma q} \\ &= \Gamma_{pq} - \lambda_{q|i,jk}\lambda_{p|}{}^i\lambda^j\lambda^k. \end{aligned} \tag{6.2}$$

The invariant Θ can now be found as before in terms of the solutions of the linear equations (6.1).

If one of the vectors, say $\lambda_{2|}{}^i$, is a solution of the parallel transport equations, we have $\gamma_{2\sigma} = 0$ from (1.2). Also, it can easily be verified that for any vector $\lambda_{3|}{}^i$ orthogonal to such a vector and to the curve,

$$\sum_\sigma e_\sigma(\gamma_{3\sigma})^2 = g_{ij}\lambda_{3|}{}^i{}_{,h}\lambda_{3|}{}^j{}_{,k}\lambda^h\lambda^k = 0,$$

* This has been proposed by Professor E. T. Whittaker as an alternative definition of spatial distance in a general space. Another definition by H. S. Ruse, *Proc. Roy. Soc. Edin.*, 52 (1932), 183, leads to this result in a general space.

and hence, from (6.2),

$$\rho_{pq} = \Gamma_{pq} \qquad (p, q = 2, 3). \tag{6.3}$$

Thus equations (4.1) are still true when only one of the vectors is given by parallel transport along the geodesic.

7. The application to an important class of spaces

We shall now consider the application of the foregoing methods to space of the common form*

$$\gamma dt^2 - \gamma^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{7.1}$$

where γ is a function of r alone. It is easily shown that, by a suitable choice of s, the equations of a typical null geodesic can be written in the form

$$\frac{dt}{ds} = \frac{1}{\gamma}, \qquad \frac{dr}{ds} = e\psi, \qquad \theta = \tfrac{1}{2}\pi, \qquad \frac{d\phi}{ds} = \frac{h}{r^2}. \tag{7.2}$$

where

$$\psi^2 = 1 - \frac{h^2\gamma}{r^2}, \qquad e = \pm 1, \tag{7.3}$$

and h is a constant. If the geodesic is through the origin, h is zero. A solution of the transport equations along this curve is

$$\lambda_{2|}{}^i \equiv (0, 0, 1/r, 0), \tag{7.4}$$

and a vector orthogonal to this vector and to the curve is

$$\lambda_{3|}{}^i \equiv (0, -eh\gamma/r, 0, \psi/r). \tag{7.5}$$

The non-vanishing components of the curvature tensor are

$$\begin{aligned} &R_{0110} = \tfrac{1}{2}\gamma'', \qquad R_{0220} = \tfrac{1}{2}r\gamma\gamma', \qquad R_{0330} = \tfrac{1}{2}r\gamma\gamma' \sin^2\theta, \\ &R_{1221} = -\tfrac{1}{2}r\frac{\gamma'}{\gamma}, \qquad R_{1331} = -\tfrac{1}{2}r\frac{\gamma'}{\gamma}\sin^2\theta, \qquad R_{2332} = -r^2(\gamma - 1)\sin^2\theta, \end{aligned} \tag{7.6}$$

and hence, from (1.4), (7.2), (7.4), and (7.5), we find

$$\begin{aligned} \Gamma_{22} &= \frac{h^2}{r^4}(\tfrac{1}{2}r\gamma' - \gamma + 1), \\ \Gamma_{23} &= 0, \\ \Gamma_{33} &= \frac{h^2}{2r^3}(r\gamma'' - \gamma'). \end{aligned} \tag{7.7}$$

* This form includes some well-known spaces in general relativity, e.g. the gravitational fields of a material particle and an electron.

If the geodesic passes through the origin, $h = 0$, and from (7.7),

$$\Gamma_{pq} = 0 \qquad (p, q = 2, 3).$$

The null geodesics through the point $s = s_0$ are therefore given by

$$z^2 = \alpha(s-s_0), \qquad z^3 = \beta(s-s_0),$$

and we have

$$\Theta = s-s_0. \tag{7.8}$$

In this case $r \pm s = \text{const.}$, and hence $\Theta = r+r_0$, $|r-r_0|$, according as the star and observer are, or are not, separated by the origin.

If $h \neq 0$, write $z^2 = rx$, $z^3 = r\psi y$. Substituting (7.7) in (4.1), and changing the independent variable from s to ϕ by (7.2), these equations become

$$\frac{d^2x}{d\phi^2} + x = 0, \tag{7.9}$$

$$\frac{d^2y}{d\phi^2} + \frac{2}{\psi}\frac{d\psi}{d\phi}\frac{dy}{d\phi} = 0. \tag{7.10}$$

Hence we have

$$z^2 = \alpha r \sin(\phi-\phi_0), \tag{7.11}$$

$$z^3 = \beta r\psi(V-V_0), \tag{7.12}$$

where

$$V = \int^{\phi} \frac{d\phi}{\psi^2},$$

and the subscript $_0$ indicates the value at the position of the star. The required invariant is now given by

$$\pm\Theta^2 = Kr^2\psi(V-V_0)\sin(\phi-\phi_0),$$

where

$$\frac{1}{K} = \lim_{s\to s_0} \frac{r^2\psi(V-V_0)\sin(\phi-\phi_0)}{(s-s_0)^2}.$$

We have

$$\frac{ds}{d\phi} = \frac{r^2}{h}, \qquad \lim_{\phi\to\phi_0} \frac{\sin(\phi-\phi_0)}{\phi-\phi_0} = 1, \qquad \lim_{\phi\to\phi_0} \frac{V-V_0}{\phi-\phi_0} = \frac{1}{\psi_0^2}.$$

Hence $K = r_0^2\psi_0/h^2$, and we have

$$\pm\Theta^2 = \frac{1}{h^2} r^2 r_0^2 \psi\psi_0 (V-V_0)\sin(\phi-\phi_0). \tag{7.13}$$

It is interesting to observe that, in certain cases, this method of defining spatial distance will give unexpected results. These would be due to the term $\sin(\phi-\phi_0)$ in the above formula, for it may be possible for the null geodesics to be sufficiently curved so that ϕ_0 and $\pi+\phi_0$ give distinct points on the curve for some values of ϕ_0. In such

a case, if the observer is placed near the point $\pi+\phi_0$, the star would appear to be very bright, and the above definition would indicate only a small spatial distance. This could quite possibly occur when light from a star passes near another heavy body, and it should be possible to find two stars, satisfying the required conditions, such that the earth is near the critical point of observation. It is seen that the above peculiarities arise from equation (7.11), which shows that it is possible for a pencil of null geodesics to converge again in one direction.

In conclusion we may remark that the above method of calculating spatial distance has been successfully applied to space of the more general form

$$ds^2 = \nu dt^2 - R^2(\gamma^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2),$$

where R is any function of t, and ν, γ are any functions of r.